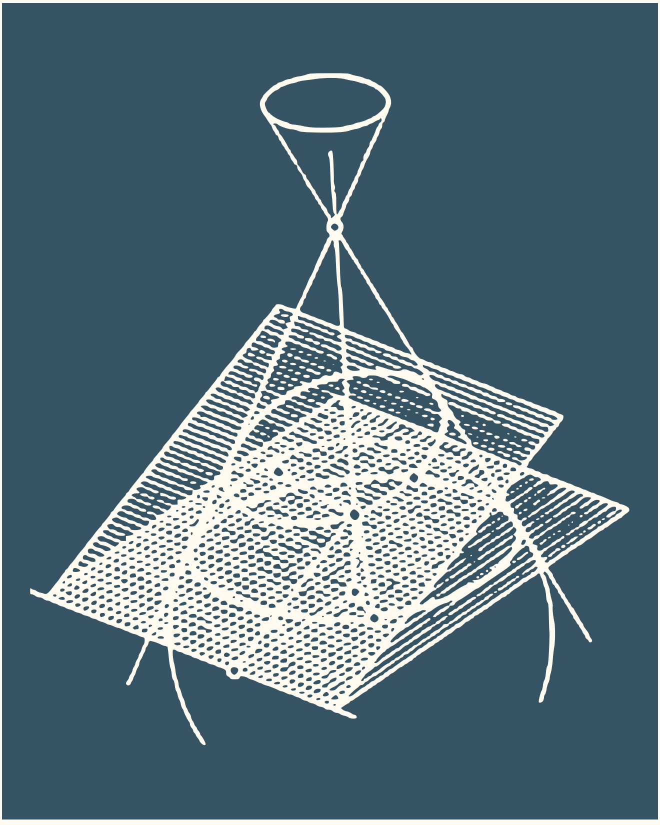
ANALYTICAL GEOMETRY

A. V. Pogorelov



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АНАЛИТИЧЕСКАЯ ГЕОМЕТРИЯ

Издательство «Наука»

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ANALYTICAL GEOMETRY

Translated
from the Russian
by
Leonid Levant

Mir Publishers
Moscow

First published 1980

На английском языке

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Introduction

Analytical geometry has no strictly defined contents. It is the method but not the subject under investigation, that constitutes the leading feature of this branch of geometry.

The essence of this method consists in that geometric objects are associated in some standard way with equations (or systems of equations) so that geometric relations of figures are expressed through properties of their equations.

For instance, in case of Cartesian coordinates any straight line in the plane is uniquely associated with a linear equation

$$ax + by + c = 0.$$

The intersection of three straight lines at one point is expressed by the condition of compatibility of a system of three equations which specify these lines.

Due to a multipurpose approach to solving various problems, the method of analytic geometry has become the leading method in geometric investigations and is widely applied in other fields of exact natural sciences, such as mechanics and physics.

Analytical geometry joined geometry with algebra and analysis —the fact which has told fruitfully on further development of these three subjects of mathematics.

The principal ideas of analytical geometry are traced back to the French mathematician, René Descartes (1596-1650), who in 1637 described the fundamentals of its method in his famous work "Geometrie".

The present book, which is a course of lectures, treats the fundamentals of the method of analytic geometry as applied to the simplest geometric objects. It is designed for the university students majoring in physics and mathematics.

Chapter 1

Rectangular Cartesian Coordinates in a Plane

Sec. 1-1. Introducing Coordinates in a Plane

Let us draw in the plane two mutually perpendicular intersecting lines Ox and Oy which are termed *coordinate axes* (Fig. 1). The point of intersection O of the two axes is called the *origin of coordinates*, or simply the *origin*. It

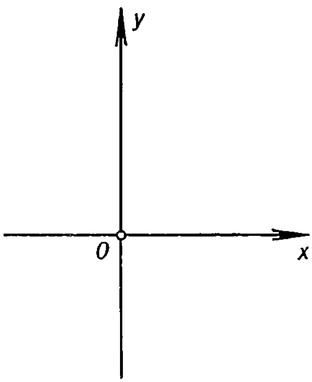


Fig. 1

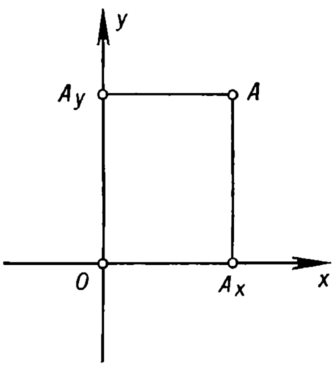


Fig. 2.

divides each of the axes into two semi-axes. One of the semi-axes is conventionally called *positive* (indicated by an arrow in the drawing), the other being *negative*.

Any point A in a plane is specified by a pair of numbers—called the *rectangular coordinates of the point* A—the *abscissa* (x) and the *ordinate* (y) according to the following rule.

Through the point A we draw a straight line parallel to the *axis of ordinates* (Oy) to intersect the *axis of abscissas*

(Ox) at some point A_x (Fig. 2). The abscissa of the point A should be understood as a number x whose absolute value is equal to the distance from O to A_x which is positive if A_x belongs to the positive semi-axis and negative if A_x belongs to the negative semi-axis. If the point A_x coincides with the origin, then we put x equal to zero.

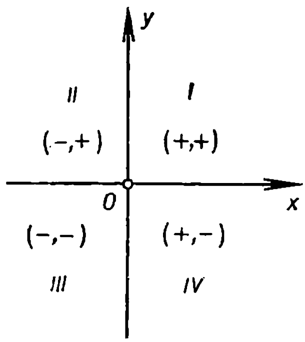


Fig. 3.

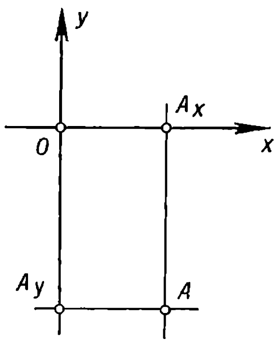


Fig. 4.

The ordinate (y) of the point A is determined in a similar way.

We shall use the following notation: A (x, y) which means that the coordinates of the point A are x (abscissa) and y (ordinate).

The coordinate axes separate the plane into four right angles termed the *quadrants* as shown in Fig. 3. Within the limits of one quadrant the signs of both coordinates remain unchanged. As we see in the figure, the quadrants are denoted and called the first, second, third, and fourth as counted anticlockwise beginning with the quadrant in which both coordinates are positive.

If a point lies on the *x-axis* (i.e. on the axis of abscissas) then its ordinate y is equal to zero; if a point lies on the *y-axis*, (i.e. on the axis of ordinates), then its abscissa x is zero. The abscissa and ordinate of the origin (i.e. of the point O) are equal to zero.

The plane on which the coordinates x and y are introduced by the above method will be called the *xy-plane*. An arbitrary point in this plane with the coordinates x and y will sometimes be denoted simply (x, y).

For an arbitrary pair of real numbers x and y there exists a unique point A in the xy-plane for which x will be its abscissa and y its ordinate.

Indeed, suppose for definiteness $x > 0$, and $y < 0$. Let us take on the positive semi-axis x a point A_x at the

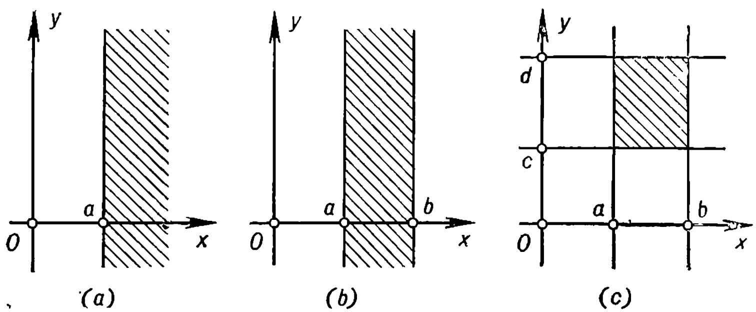


Fig. 5.

distance x from the origin O, and a point A_y on the negative semi-axis y at the distance $|y|$ from O. We then draw through the points A_x and A_y straight lines parallel to the axes y and x, respectively (Fig. 4). These lines will intersect at a point A whose abscissa is obviously x, and ordinate is y. In other cases ($x < 0$, $y > 0$; $x > 0$, $y > 0$ and $x < 0$, $y < 0$) the proof is analogous.

Let us consider several important cases of analytical representation of domains on the xy-plane with the aid of inequalities. A set of points of the xy-plane for which $x > a$ is a half-plane bounded by a straight line passing through the point $(a, 0)$ parallel to the axis of ordinates (Fig. 5, a). A set of points for which $a < x < b$ represents the intersection (i.e. the common portion) of the half-planes specified by the inequalities $a < x$ and $x < b$. Thus, this set is a band between the straight lines parallel to the y-axis and passing through the points $(a, 0)$ and $(b, 0)$ (Fig. 5, b). A set of points for which $a < x < b$, $c < y < d$ is a rectangle with vertices at points (a, c), (a, d), (b, c), (b, d) (Fig. 5, c).

In conclusion, let us solve the following problem: *Find the area of a triangle with vertices at points $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, $A_3(x_3, y_3)$*. Let the triangle be located relative to the coordinate system as is shown in Fig. 6. In this position its area is equal to the difference between the area of the trapezium $B_1A_1A_3B_3$ and the sum of the areas of the trapezia $B_1A_1A_2B_2$ and $B_2A_2A_3B_3$.

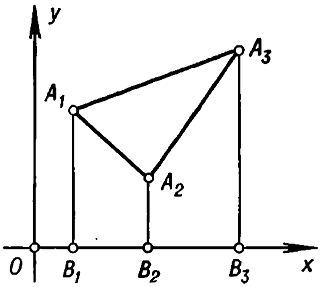


Fig. 6.

The bases of the trapezium $B_1A_1A_3B_3$ are equal to y_1 and y_3, its altitude being equal to x_3-x_1. Therefore, the area of the trapezium

$$S(B_1A_1A_3B_3)=\frac{1}{2}(y_3+y_1)(x_3-x_1).$$

The areas of two other trapezia are found analogously:

$$S(B_1A_1A_2B_2)=\frac{1}{2}(y_2+y_1)(x_2-x_1),$$

$$S(B_2A_2A_3B_3)=\frac{1}{2}(y_3+y_2)(x_3-x_2).$$

The area of the triangle $A_1A_2A_3$:

$$S(A_1A_2A_3)=\frac{1}{2}(y_3+y_1)(x_3-x_1)-$$
$$-\frac{1}{2}(y_2+y_1)(x_2-x_1)-\frac{1}{2}(y_3+y_2)(x_3-x_2)=$$
$$=\frac{1}{2}(x_2y_3-y_3x_1+x_1y_2-y_2x_3+x_3y_1-y_1x_2).$$

This formula can be rewritten in a more convenient form:

$$S(A_1A_2A_3)=\frac{1}{2}\{(y_3-y_1)(x_2-x_1)-(y_2-y_1)(x_3-x_1)\}.$$

Though the above formula for computing the area of the triangle has been derived for a particular location of the triangle relative to the coordinate system, it yields a correct result (to within a digit) for any position of the triangle. This will be proved later on (in Sec. 2-5).

EXERCISES

1. What is the location of the points of the xy-plane for which (a) $|x| = a$, (b) $|x| = |y|$?
2. What is the location of the points of the xy-plane for which (a) $|x| < a$, (b) $|x| < a$, $|y| < b$?
3. Find the coordinates of a point symmetrical to the point $A(x, y)$ about the x-axis (y-axis, the origin).
4. Find the coordinates of a point symmetrical to the point $A(x, y)$ about the bisector of the first (second) quadrant.
5. How will the coordinates of the point $A(x, y)$ change if the y-axis is taken for the x-axis, and vice versa?
6. How will the coordinates of the point $A(x, y)$ change if the origin is displaced into the point $A_0(x_0, y_0)$ without changing the directions of the coordinate axes?
7. Find the coordinates of the mid-points of the sides of a square taking its diagonals for the coordinate axes.
8. It is known that three points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear. How can one find out which of these points is situated between the other two?

Sec. 1-2. The Distance Between Points

Let there be given on the xy-plane two points: A_1 with the coordinates x_1, y_1 and A_2 with the coordinates x_2, y_2. *It is required to express the distance between the points A_1 and A_2 in terms of their coordinates.*

Suppose $x_1 \neq x_2$ and $y_1 \neq y_2$. Through the points A_1 and A_2 we draw straight lines parallel to the coordinate axes (Fig. 7). The distance between the points A and A_1 is equal to $|y_1 - y_2|$, and the distance between the points A and A_2 is equal to $|x_1 - x_2|$. Applying the Pythagorean theorem to the right-angled triangle A_1AA_2, we get

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = d^2, \qquad (*)$$

where d is the distance between the points A_1 and A_2.

Though the formula (*) for determining the distance between points has been derived by us proceeding from the assumption that $x_1 \neq x_2$, $y_1 \neq y_2$, it remains true for other cases as well. Indeed, for $x_1 = x_2$, $y_1 \neq y_2$ d is equal to $|y_1 - y_2|$ (Fig. 8). The same result is obtained using the formula (*). For $x_1 \neq x_2$, $y_1 = y_2$ we get a simi-

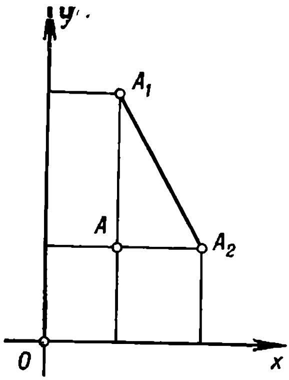


Fig. 7.

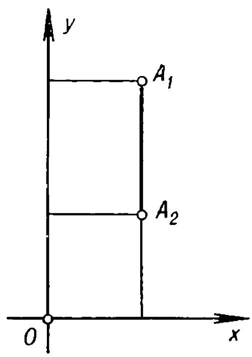


Fig. 8.

lar result. If $x_1 = x_2$, $y_1 = y_2$ the points A_1 and A_2 coincide and the formula (*) yields $d = 0$.

As an exercise, *let us find the coordinates of the centre of a circle circumscribed about a triangle with the vertices* (x_1, y_1), (x_2, y_2), and (x_3, y_3). 3

Let (x, y) be the centre of the circumcircle. Since it is equidistant from the vertices of the triangle, we derive the following equations for the required coordinates of the centre of the circle (x and y). Thus, we have

$$(x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2,$$
$$(x - x_1)^2 + (y - y_1)^2 = (x - x_3)^2 + (y - y_3)^2,$$

or after obvious transformations

$$2(x_2 - x_1)x + 2(y_2 - y_1)y = x_2^2 + y_2^2 - x_1^2 - y_1^2,$$
$$2(x_3 - x_1)x + 2(y_3 - y_1)y = x_3^2 + y_3^2 - x_1^2 - y_1^2.$$

Thus, we have a system of two linear equations for determining the unknowns x and y.

EXERCISES

1. Find on the x-axis the coordinates of a point equidistant from the two given points A (x_1, y_1), and B (x_2, y_2). Consider the case A $(0, a)$, B $(b, 0)$.

2. Given the coordinates of two vertices A and B of an equilateral triangle ABC. How to find the coordinates of the third vertex? Consider the case A $(0, a)$, B $(a, 0)$.

3. Given the coordinates of two adjacent vertices A and B of a square $ABCD$. How are the coordinates of the remaining vertices found? Consider the case A $(a, 0)$, B $(0, b)$.

4. What condition must be satisfied by the coordinates of the vertices of a triangle ABC so as to obtain a right-angled triangle with a right angle at the vertex C?

5. What condition must be satisfied by the coordinates of the vertices of a triangle ABC so that the angle A exceeds the angle B?

6. A quadrilateral $ABCD$ is specified by the coordinates of its vertices. How to find out whether or not is it inscribed in a circle?

7. Prove that for any real a, a_1, a_2, b, b_1, b_2 there exists the following inequality

$$\sqrt{(a_1-a)^2+(b_1-b)^2}+\sqrt{(a_2-a)^2+(b_2-b)^2} \geqslant \geqslant \sqrt{(a_1-a_2)^2+(b_1-b_2)^2}.$$

To what geometrical fact does it correspond?

Sec. 1-3. Dividing a Line Segment in a Given Ratio

Let there be given two different points on the xy-plane: A_1 (x_1, y_1) and A_2 (x_2, y_2). *Find the coordinates x and y of the point A which divides the segment A_1A_2 in the ratio $\lambda_1 : \lambda_2$.*

Suppose the segment A_1A_2 is not parallel to the x-axis. Projecting the points A_1, A, A_2 on the y-axis, we have (Fig. 9)

$$\frac{A_1A}{AA_2}=\frac{\bar{A}_1\bar{A}}{\bar{A}\bar{A}_2}=\frac{\lambda_1}{\lambda_2}.$$

Since the points $\bar{A}_1$, $\bar{A}_2$, $\bar{A}$ have the same ordinates as the points A_1, A_2, A, respectively, we get

$$\bar{A}_1\bar{A} = |y_1 - y|, \quad \bar{A}\bar{A}_2 = |y - y_2|.$$

Consequently,

$$\frac{|y_1-y|}{|y-y_2|} = \frac{\lambda_1}{\lambda_2}.$$

Since the point $\bar{A}$ lies between $\bar{A}_1$ and $\bar{A}_2$, $y_1 - y$ and $y - y_2$ have the same sign. Therefore

$$\frac{|y_1-y|}{|y-y_2|} = \frac{y_1-y}{y-y_2} = \frac{\lambda_1}{\lambda_2}.$$

Whence we find

$$y = \frac{\lambda_2 y_1 + \lambda_1 y_2}{\lambda_1 + \lambda_2}. \qquad (*)$$

If the segment A_1A_2 is parallel to the x-axis, then

$$y_1 = y_2 = y.$$

The same result is yielded by the formula $(*)$ which is thus true for any positions of the points A_1 and A_2.

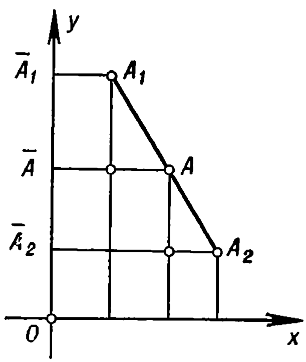


Fig. 9.

The abscissa of the point A is found analogously. For it we get the formula

$$x = \frac{\lambda_2 x_1 + \lambda_1 x_2}{\lambda_1 + \lambda_2}.$$

We put $\frac{\lambda_1}{\lambda_1+\lambda_2} = t$. Then $\frac{\lambda_2}{\lambda_1+\lambda_2} = 1 - t$.

Consequently, the coordinates of any point C of a segment with the end-points A (x_1, y_1) and B (x_2, y_2) may be represented as follows

$$x = (1-t)\,x_1 + tx_2, \quad y = (1-t)\,y_1 + ty_2, \quad 0 \leqslant t \leqslant 1.$$

Let us find the location of points C (x, y) for $t < 0$ and $t > 1$. To do this in case of $t < 0$ we solve our formulas with respect to x_1, y_1. We get

$$x_1=\frac{1\cdot x+(-t)\,x_2}{1-t},$$

$$y_1=\frac{1\cdot y+(-t)\,y_2}{1-t}.$$

Hence, it is clear that the point A (x_1, y_1) is situated on the line segment CB and divides this segment in the ratio $(-t)$: 1. Thus, for $t<0$ our formulas yield the coordinates of the point lying on the extension of the segment AB beyond the point A. It is proved in a similar way that for $t>1$ the formulas yield the coordinates of the point located on the extension of the segment AB beyond the point B.

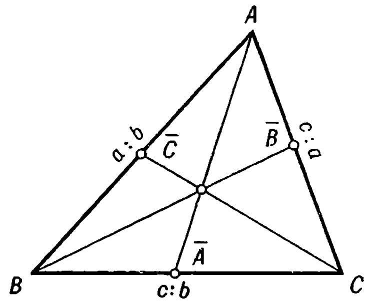


Fig. 10.

As an exercise, let us prove Ceva's theorem from elementary geometry. It states: *If the sides of a triangle are divided in the ratio* $a:b$, $c:a$, $b:c$, *taken in order of moving round the triangle* (see Fig. 10), *then the segments joining the vertices of the triangle to the points of division of the opposite sides intersect in one point.*

Let A (x_1, y_1), B (x_2, y_2), and C (x_3, y_3) be the vertices of the triangle and $\bar{A}$, $\bar{B}$, $\bar{C}$ the points of division of the opposite sides (Fig. 10). The coordinates of the point $\bar{A}$ are:

$$x=\frac{bx_2+cx_3}{b+c}, \quad y=\frac{by_2+cy_3}{b+c}.$$

Let us divide the line segment $A\bar{A}$ in the ratio $(b+c):a$. Then the coordinates of the point of division will be

$$x=\frac{ax_1+bx_2+cx_3}{a+b+c},$$

$$y=\frac{ay_1+by_2+cy_3}{a+b+c}.$$

If the segment $B\bar{B}$ is divided in the ratio $(a + c) : b$, then we get the same coordinates of the point of division. The same coordinates are obtained when dividing the segment $C\bar{C}$ in the ratio $(a + b) : c$. Hence, the segments $A\bar{A}$, $B\bar{B}$, and $C\bar{C}$ have a point in common, which was required to be proved.

Let us note here that the theorems of elementary geometry on intersecting medians, bisectors, and altitudes in the triangle are particular cases of Ceva's theorem.

EXERCISES

1. Given the coordinates of three vertices of a parallelogram: (x_1, y_1), (x_2, y_2), and (x_3, y_3). Find the coordinates of the fourth vertex and the centroid.

2. Given the coordinates of the vertices of a triangle: (x_1, y_1), (x_2, y_2), and (x_3, y_3). Find the coordinates of the point of intersection of the medians.

3. Given the coordinates of the mid-points of the sides of a triangle (x_1, y_1), (x_2, y_2), and (x_3, y_3). Find the coordinates of its vertices.

4. Given a triangle with the vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3). Find the coordinates of the vertices of a homothetic triangle with the ratio of similitude λ and the centre of similitude at point (x_0, y_0).

5. Point A is said *to divide* the line segment A_1A_2 *externally* in the ratio $\lambda_1 : \lambda_2$ if this point lies on a straight line joining the points A_1 and A_2 outside the segment A_1A_2 and the ratio of its distances from the points A_1 and A_2 is equal to $\lambda_1 : \lambda_2$. Show that the coordinates of the point A are expressed in terms of the coordinates (x_1, y_1), (x_2, y_2) of the points A_1 and A_2 by the formulas

$$x = \frac{\lambda_2 x_1 - \lambda_1 x_2}{\lambda_2 - \lambda_1}, \quad y = \frac{\lambda_2 y_1 - \lambda_1 y_2}{\lambda_2 - \lambda_1}.$$

6. Two line segments are specified by the coordinates of their end-points. How can we find out, without using a drawing, whether the segments intersect or not?

7. *The centre of gravity of two masses* μ_1 and μ_2 situated at points A_1 (x_1, y_1) and A_2 (x_2, y_2) is defined as a point A which divides the segment A_1A_2 in the ratio $\mu_2 : \mu_1$.

Thus, its coordinates are:

$$x = \frac{\mu_1 x_1 + \mu_2 x_2}{\mu_1 + \mu_2}, \quad y = \frac{\mu_1 y_1 + \mu_2 y_2}{\mu_1 + \mu_2}.$$

The centre of gravity of n masses μ_i situated at points A_i is determined by induction. Indeed, if A'_n is the centre of gravity of the first $n - 1$ masses, then the centre of gravity of all n masses is determined as the centre of gravity of two masses: μ_n located at point A_n, and $\mu_1 + \ldots + \mu_{n-1}$, situated at point A'_n. We then derive the formulas for the coordinates of the centre of gravity of the masses μ_i situated at points A_i (x_i, y_i):

$$x = \frac{\mu_1 x_1 + \ldots + \mu_n x_n}{\mu_1 + \ldots + \mu_n}, \quad y = \frac{\mu_1 y_1 + \ldots + \mu_n y_n}{\mu_1 + \ldots + \mu_n}.$$

Sec. 1-4. The Notion of the Equation of a Curve. The Equation of a Circle

Let there be given a curve on the xy-plane (Fig. 11). The equation $\varphi(x, y) = 0$ is called the *equation of a curve in the implicit form* if it is satisfied by the coordinates

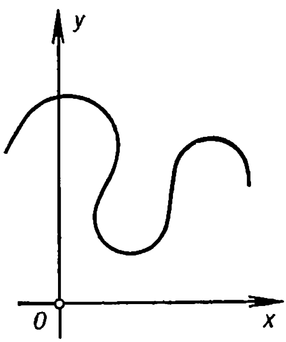


Fig. 11.

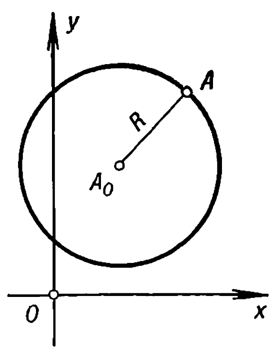


Fig. 12

(x, y) of any point of this curve and any pair of numbers x, y, satisfying the equation $\varphi(x, y) = 0$ represents the coordinates of a point on the curve. As is obvious, a curve is defined by its equation, therefore we may speak of representing a curve by its equation.

In analytic geometry two problems are often considered: (1) given the geometrical properties of a curve, form its equation; (2) given the equation of a curve, find out its geometrical properties. Let us consider these problems as applied to the circle which is the simplest curve.

Suppose that A_0 (x_0, y_0) is an arbitrary point of the xy-plane, and R is any positive number. Let us form the equation of a circle with centre A_0 and radius R (Fig.12).

Let A (x, y) be an arbitrary point of the circle. Its distance from the centre A_0 is equal to R. According to Sec. 1-2, the square of the distance of the point A from A_0 is equal to $(x-x_0)^2+(y-y_0)^2$. Thus, the coordinates x, y of any point A of the circle satisfy the equation

$$(x-x_0)^2+(y-y_0)^2-R^2=0. \qquad (*)$$

Conversely, any point A whose coordinates satisfy the equation (*) belongs to the circle, since its distance from A_0 is equal to R.

In conformity with the above definition, the equation (*) *is an equation of a circle with centre A_0 and radius R.*

We now consider the second problem for the curve given by the equation

$$x^2+y^2+2ax+2by+c=0 \quad (a^2+b^2-c>0).$$

This equation can be rewritten in the following equivalent form:

$$(x+a)^2+(y+b)^2-\left(\sqrt{a^2+b^2-c}\right)^2=0.$$

Whence it is seen that any point (x, y) of the curve i found at one and the same distance equal to $\sqrt{a^2+b^2-c}$ from the point $(-a, -b)$, and, hence, the *curve is a circle with centre $(-a, -b)$ and radius $\sqrt{a^2+b^2-c}$.*

Let us consider the following problem as an example illustrating the application of the method of analytic geometry: *Find the locus of points in a plane the ratio of whose distances from two given points A and B is constant and is equal to $k \neq 1$.* (The *locus* is defined as a figure which consists of all the points possessing the given geometrical property. In the case under consideration we speak of a set of all the points in the plane for which the ratio of the distances from the two given points A and B is constant).

Suppose that $2a$ is the distance between the points A and B. We then introduce a rectangular Cartesian coordinate system on the plane taking the straight line AB for the x-axis and the midpoint of the segment AB for the origin. Let, for definiteness, the point A be situated on the positive semi-axis x. The coordinates of the point A will then be: $x = a$, $y = 0$, and the coordinates of the point B will be: $x = -a$, $y = 0$. Let (x, y) be an arbitrary point of the locus. The squares of its distances from the points A and B are respectively equal to $(x - a)^2 + y^2$ and $(x + a)^2 + y^2$. The equation of the locus is

$$\frac{(x-a)^2+y^2}{(x+a)^2+y^2} = k^2,$$

or

$$x^2+y^2+\frac{2(k^2+1)}{k^2-1}ax+a^2=0.$$

The locus represents a circle (Apollonius' circle).

Let us consider another problem as an example of forming the equation of a circle. Given are the equations of two circles

$$x^2 + y^2 + a_1x + b_1y + c_1 = 0,$$
$$x^2 + y^2 + a_2x + b_2y + c_2 = 0$$

and a point A (x_1, y_1). *Form the equation of a circle passing through the points of intersection of the given circles and the point A.*

The usual solution of this problem consists in that we determine the points of intersection of the given circles and then find the equation of the circle passing through the found points of intersection and the given point A. Let us consider a more "economical" way of solving this problem.

For any λ and μ the equation

$$\lambda (x^2 + y^2 + a_1x + b_1y + c_1) + \mu (x^2+y^2+a_2x+ + b_2y + c_2) = 0,$$

represents a circle if $\lambda + \mu \neq 0$. This circle passes through the points of intersection of the given circles, since the coordinates of these points reduce to zero both terms of the left-hand side of the equation. If we select λ, μ so that the coordinates of the point A satisfy this equation, then we shall get the required circle. As is obvious, the following choice will suit

$$\lambda = x_1^2 + y_1^2 + a_2x_1 + b_2y_1 + c_2,$$
$$-\mu = x_1^2 + y_1^2 + a_1x_1 + b_1y_1 + c_1.$$

Geometrically, it is clear that the problem has no solution if the point A lies on the straight line joining the points of intersection of the given circles. Analytically, it is expressed by the fact that the equation obtained does not contain the term $x^2 + y^2$.

EXERCISES

1. What peculiarities in the position of the circle

$$x^2 + y^2 + 2ax + 2by + c = 0 \quad (a^2 + b^2 - c > 0)$$

relative to the coordinate system take place if

(1) $a = 0$; (2) $b = 0$; (3) $c = 0$; (4) $a = 0$, $b = 0$; (5) $a = 0$, $c = 0$; (6) $b = 0$, $c = 0$?

2. Show that if we substitute in the left-hand member of the equation of a circle the coordinates of any point lying outside the circle, then the square of the length of a tangent drawn from this point to the circle is obtained.

3. The *power of a point* A with reference to a circle is defined as the product of the segments of a secant drawn through the point A taken with plus for outside points and with minus for inside points. Show that the left-hand member of the equation of a circle $x^2 + y^2 + 2ax + 2by + c = 0$ gives the power of this point with reference to a circle when the coordinates of an arbitrary point are substituted in it.

4. Form the equation of the locus of points of the xy-plane the sum of whose distances from two given points F_1 $(c, 0)$ and F_2 $(-c, 0)$ is constant and is equal to $2a$ (the *ellipse*). Show that the equation is reduced to the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2 - c^2$.

5. Form the equation of the locus of points of the xy-plane the difference of whose distances from two given points F_1 $(c, 0)$ and F_2 $(-c, 0)$ is constant and is equal to $2a$ (the *hyperbola*). Show that the equation is reduced to the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = c^2 - a^2$.

6. Form the equation of the locus of points of the xy-plane which are equidistant from the point F $(0, p)$ and the x-axis (the *parabola*).

Sec. 1-5. The Equation of a Curve Represented Parametrically

Suppose a point A moves along a curve, and by the time t its coordinates are: $x = \varphi(t)$ and $y = \psi(t)$. A system of equations

$$x = \varphi(t), \quad y = \psi(t),$$

specifying the coordinates of an arbitrary point on the curve as functions of the parameter t is called the *equation of a curve in parametric form*.

The parameter t is not necessarily time, it may be any other quantity characterizing the position of a point on the curve.

Let us now form the equation of a circle in parametric form.

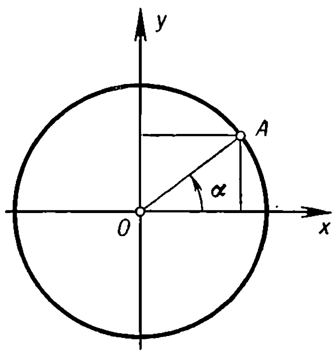


Fig. 13.

Suppose the centre of a circle is situated at the origin, and the radius is equal to R. We shall characterize the position of point A on the circle by the angle α formed by the radius OA with the positive semi-axis x (Fig. 13). As is obvious, the coordinates of the point A are equal to $R \cos \alpha$, $R \sin \alpha$, and, consequently, the equation of the circle has such a form:

$$x = R \cos \alpha, \quad y = R \sin \alpha,$$

Having an equation of a curve in parametric form:

$$x = \varphi(t), \ y = \psi(t), \tag{*}$$

we can obtain its equation in implicit form:

$$f(x, y) = 0.$$

To this effect it is sufficient to eliminate the parameter t from the equations (*), finding it from one equation and substituting into the other, or using another method.

For instance, to get the equation of a circle represented by equations in parametric form (i.e. implicitly) it is sufficient to square both equalities and add them termwise. We then obtain the familiar equation $x^2 + y^2 = R^2$.

The elimination of the parameter from the equations of a curve represented parametrically not always yields an equation in implicit form in the sense of the above definition. It may turn out that it is satisfied by the points not belonging to the curve. In this connection let us consider two examples.

Suppose a curve γ is given by the equations in parametric form

$$x = a \cos t, \ y = b \sin t, \ 0 \leqslant t < 2\pi.$$

Dividing these equations by a and b, respectively, squaring and adding them termwise, we get the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This equation is obviously satisfied by all the points belonging to the curve γ. Conversely, if the point (x, y) satisfies this equation, then there can be found an angle t for which $x/a = \cos t$, $y/b = \sin t$, and, consequently, any point of the plane which satisfies this equation, belongs to the curve γ.

Let now a curve γ be represented by the following equations

$$x = a \cosh t, \ y = b \sinh t, \ -\infty < t < +\infty,$$

where

$$\cosh t = (e^t + e^{-t})/2, \ \sinh t = (e^t - e^{-t})/2.$$

Dividing these equations by a and b, respectively, and then squaring them and subtracting termwise, we get the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

The points of the curve γ satisfy this equation. But not any point which satisfies the equation belongs to γ. Let us, for instance, consider the point $(-a, 0)$. We see that it satisfies the equation, but does not belong to the curve, since on the curve γ $a \cosh t \neq -a$.

Sometimes the equation of a curve represented in implicit form is understood in a wider way. One does not require that any point satisfying the equation, belongs to the curve.

EXERCISES

1. Show that the following equations in parametric form

$$x = R\cos t + a, \quad y = R\sin t + b$$

represent a circle of radius R with centre at point (a, b).

2. Form the equation of a curve described by a point on the line segment of length a when the end-points of the

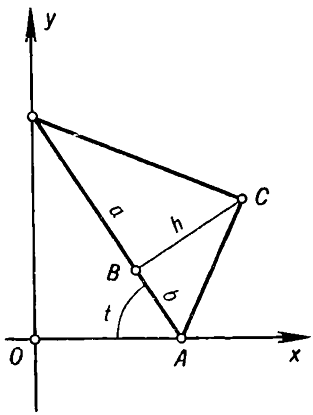


Fig. 14.

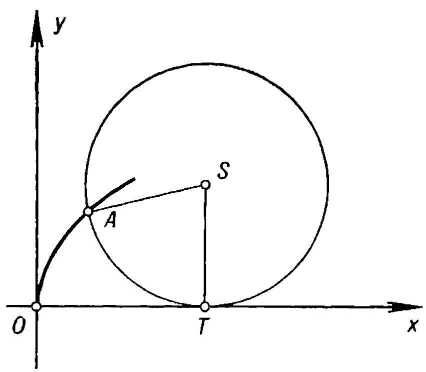


Fig. 15.

segment slide along the coordinate axes (the segment is divided by this point in the ratio $\lambda : \mu$). Take the angle formed by the segment with the x-axis for the parameter. What is the shape of the curve if $\lambda : \mu = 1$?

3. A triangle slides along the coordinate axes with two of its vertices. Form the equation of the curve described by the third vertex (Fig. 14).

4. Form the equation of the curve described by a point on a circle of radius R which rolls along the x-axis (Fig. 15). For the parameter take the path s covered by the centre of the circle and suppose that at the initial moment ($s = 0$) point A coincides with the origin.

5. A curve is given by the equation

$$ax^2 + bxy + cy^2 + dx + ey = 0.$$

Show that, by introducing the parameter $t = y/x$, we can obtain the following equations of this curve in parametric form:

$$x = -\frac{d+et}{a+bt+ct^2},$$

$$y = -\frac{dt+et^2}{a+bt+ct^2}.$$

Sec. 1-6. The Points of Intersection of Curves

Let there be given two curves in the xy-plane: the curve γ_1 represented by the equation

$$f_1(x, y) = 0,$$

and the curve γ_2 specified by the equation

$$f_2(x, y) = 0.$$

We now find the points of intersection of the curves γ_1 and γ_2, i.e. the coordinates of these points. Let $A(x, y)$ be the point of intersection of the curves γ_1 and γ_2. Since the point A lies on the curve γ_1, its coordinates satisfy the equation $f_1(x, y) = 0$. Also, since the point A lies on the curve γ_2, its coordinates satisfy the equation $f_2(x, y) = 0$. Thus, the coordinates of any point of intersection of the curves γ_1 and γ_2 satisfy the system of equations

$$f_1(x, y) = 0, \quad f_2(x, y) = 0.$$

Conversely, any real solution of this system of equations yields the coordinates of one of the points of intersection of the curves.

If the curve γ_1 is represented by the equation

$$f_1(x, y) = 0.$$

and the curve γ_2 is given by the equations in parametric form

$$x = \varphi(t), \quad y = \psi(t),$$

then the coordinates x, y of the points of intersection satisfy a system of three equations

$$f_1(x, y) = 0, \quad x = \varphi(t) \quad y = \psi(t).$$

If both curves are represented parametrically

$$\gamma_1 : x = \varphi_1(t), \quad y = \psi_1(t);$$
$$\gamma_2 : x = \varphi_2(\tau), \quad y = \psi_2(\tau),$$

then the coordinates x, y of the points of intersection satisfy the following system of four equations:

$$x = \varphi_1(t), \; y = \psi_1(t),$$
$$x = \varphi_2(\tau), \; y = \psi_2(\tau).$$

Example. Find the points of intersection of the circles

$$x^2 + y^2 = 2ax, \quad x^2 + y^2 = 2by.$$

Subtracting the equations termwise, we find $ax = by$. Substituting $y = ax/b$ in the first equation, we get

$$\left(1+\frac{a^2}{b^2}\right)x^2 - 2ax = 0.$$

Whence

$$x_1 = 0, \quad x_2 = \frac{2ab^2}{a^2+b^2},$$

the corresponding ordinates being

$$y_1 = 0, \quad y_2 = \frac{2ba^2}{a^2+b^2}.$$

The required points of intersection are $(0, 0)$ and $\left(\frac{2ab^2}{a^2+b^2}, \frac{2ba^2}{a^2+b^2}\right)$.

Let us consider another example illustrating the intersection of curves. Suppose two curves (γ_1 and γ_2) are given. The curve γ_1 is represented by an equation in implicit form

$$f(x, y) = 0,$$

where $f(x, y)$ is a polynomial of degree not exceeding n. The curve γ_2 is specified by two equations in parametric form

$$x = \varphi(t), \quad y = \psi(t),$$

where $\varphi(t)$ and $\psi(t)$ are polynomials of degree not exceeding m. Let the curves γ_1 and γ_2 have more than mn points of intersection. We are going to show that the curve γ_2 lies entirely on the curve γ_1 in a sense that all of its points satisfy the equation

$$f(x, y) = 0.$$

Indeed, the algebraic equation $f(\varphi(t), \psi(t)) = 0$ has a degree not exceeding mn and has more than mn roots. As is known from algebra, such an equation is an identity, i.e. it is satisfied for any t. This means that any point of the curve γ_2 satisfies the equation $f(x, y) = 0$, which was required to be proved.

EXERCISES

1. What condition must be satisfied by the coefficients of the equation of a circle

$$x^2 + y^2 + 2ax + 2by + c = 0$$

so that the circle (a) does not intersect the x-axis; (*b*) intersects the x-axis at two points; (c) touches the x-axis?

2. What condition must be satisfied by the coefficients of the following equations of circles

$$x^2 + y^2 + 2a_1x + 2b_1y + c_1 = 0,$$
$$x^2 + y^2 + 2a_2x + 2b_2y + c_2 = 0,$$

so that the circles (a) intersect; (b) touch each other?

3. Find the points of intersection of the two circles:

(a) $x^2 + y^2 = 1$;

(b) $x = \cos t + 1, \quad y = \sin t.$

4. Find the points of intersection of the two curves represented parametrically

$$\left.\begin{aligned} x &= s^2+1, \\ y &= s, \end{aligned}\right\} \quad \left.\begin{aligned} x &= t^2, \\ y &= t+1. \end{aligned}\right\}$$

5. Show that the points of intersection of the curves

$$ax^2+by^2=c, \quad Ax^6+By^6=C$$

are situated symmetrically about the coordinate axes.

Chapter 2

The Straight Line

Sec. 2-1. The General Equation of a Straight Line

The straight line is the simplest and most widely used line.

We shall now show that *any straight line has an equation of the form*

$$ax + by + c = 0, \qquad (*)$$

where a, b, c *are constant*. And conversely, *if* a *and* b *are not both zero, then there exists a straight line for which* $(*)$ *is its equation.*

Let $A_1\ (a_1,\ b_1)$ and $A_2\ (a_2,\ b_2)$ be two different points situated symmetrically about a given straight line (Fig. 16). Then any point $A\ (x,\ y)$ on this line is equidistant from the points A_1 and A_2. And conversely, any point A which is equidistant from A_1 and A_2 belongs to the straight line. Hence, the equation of a straight line is

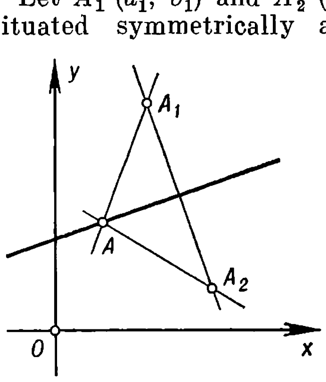


Fig. 16.

$$(x - a_1)^2 + (y - b_1)^2 = (x - a_2)^2 + (y - b_2)^2.$$

Transposing all terms of the equation to the left-hand side, removing the squared parentheses, and carrying out obvious simplifications, we get

$$2\,(a_2 - a_1)\,x + 2\,(b_2 - b_1)\,y + (a_1^2 + b_1^2 - a_2^2 - b_2^2) = 0.$$

Thus, the first part of the statement is proved.

We now shall prove the second part. Let B_1 and B_2 be two different points of the xy-plane whose coordinates satisfy the equation (*). Suppose

$$a_1x + b_1y + c_1 = 0$$

is the equation of the straight line B_1B_2. The system of equations

$$\left.\begin{aligned} ax + by + c &= 0, \\ a_1x + b_1y + c_1 &= 0 \end{aligned}\right\} \qquad (**)$$

is compatible, it is *a fortiori* satisfied by the coordinates of the point B_1, as well as of B_2.

Since the points B_1 and B_2 are different, they differ in at least one coordinate, say $y_1 \neq y_2$. Multiplying the first equation of (**) by a_1 and the second one by a, and subtracting termwise, we get

$$(ba_1 - ab_1)\, y + (ca_1 - ac_1) = 0.$$

This equation as a corollary of the equations (**) is satisfied when $y = y_1$ and $y = y_2$. But it is possible only if

$$ba_1 - ab_1 = 0, \; ca_1 - ac_1 = 0.$$

Hence it follows that

$$\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1},$$

which means that the equations (**) are equivalent. The second part of the statement is also proved.

As was shown in Sec. 1-3, the points of a straight line passing through (x_1, y_1) and (x_2, y_2) allow the following representation

$$x = (1 - t)\, x_1 + tx_2, \quad y = (1 - t)\, y_1 + ty_2.$$

Whence it follows that *any straight line allows a parametric representation by equations of the form*

$$x = at + b, \quad y = ct + d, \quad -\infty < t < \infty.$$

Conversely, any such system of equations may be considered as equations of a straight line in parametric form if a

and c are not both equal to zero. This straight line is represented by the equation in implicit form

$$(x - b)\,c - (y - d)\,a = 0.$$

EXERCISES

1. Show that the equation

$$a^2x^2 + 2abxy + b^2y^2 - c^2 = 0$$

represents a pair of straight lines. Find the equations representing each line separately.

2. A curve γ is represented by the equation $\omega(x, y) = 0$, where ω is a polynomial of degree n with respect to x and y. Show that if the curve γ has more than n points of intersection with a straight line, then it contains this line entirely.

3. Show that if the coefficients of the equations of two different straight lines

$$ax + by + c = 0, \quad Ax + By + C = 0$$

satisfy the condition

$$Ab - aB = 0,$$

then the straight lines are parallel to each other, i.e. they do not intersect.

4. The *radical axis* of two circles is the locus of points whose powers with respect to the circles are equal (see Exercise 3 of Sec. 1-4). Show that the radical axis is a straight line. If the circles intersect, then it passes through the points of intersection.

5. Show that the locus of points in the plane the difference of whose distances from two given points is constant is a straight line.

6. *Inversion* of a point with respect to a circle consists in finding the point on the radial line through the given point such that the product of the distances of the two points from the centre of the circle is equal to the square of the radius.

Consider a fixed circle, centre O and radius R, and any point A. The point A' on the ray OA such that $OA \cdot OA' = R^2$ is called the inverse of A with respect to the fixed circle. The circle is called the circle of inversion, its centre is the centre of inversion, its radius is called the radius of inversion, and R^2 is called the constant of inversion.

Suppose O is at the origin. Show that the coordinates of the point A' are expressed in terms of the coordinates of the point A by the following formulas

$$x' = \frac{R^2 x}{x^2 + y^2}, \quad y' = \frac{R^2 y}{x^2 + y^2}.$$

7. Show that the inverse of a circle is a circle or a straight line (when a straight line?).

8. Find the coordinates of a point A^* which is symmetrical to the point $A\ (x_0, y_0)$ about the straight line $ax + by + c = 0$.

9. Show that three points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Sec. 2-2. Particular Cases of the Equation of a Straight Line

Let us find out the peculiarities which happen in the location of a straight line relative to the coordinate system if its equation $ax + by + c = 0$ is of a particular form.

1. $a = 0$. In this case the equation of a straight line can be rewritten as follows

$$y = -\frac{c}{b}.$$

Thus, all points belonging to the straight line have one and the same ordinate $(-c/b)$, and, consequently, the *line is parallel to the x-axis* (Fig. 17, *a*). In particular, *if* $c = 0$, *then the straight line coincides with the x-axis.*

2. $b = 0$. This case is considered in a similar way. *The straight line is parallel to the y-axis* (Fig. 17, *b*) *and coincides with it if c is also zero.*

3. $c = 0$. *The straight line passes through the origin*, since the coordinates of the latter (0, 0) satisfy the equation of the straight line (Fig. 17, *c*).

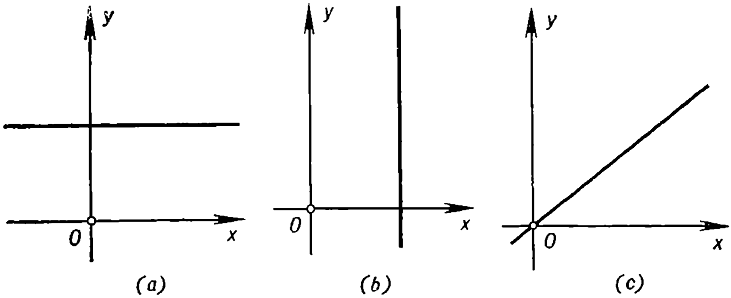


Fig. 17.

4. Suppose all the coefficients of the equation of the straight line are non-zero (i.e. the line does not pass through the origin and is not parallel to the coordinate axes). Then, multiplying the equation by $1/c$ and putting $-c/a = \alpha$, $-c/b = \beta$, we reduce it to the form

$$\frac{x}{\alpha}+\frac{y}{\beta}=1. \qquad (*)$$

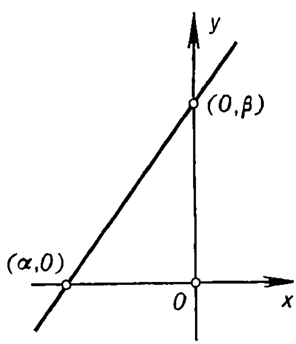


Fig. 18.

The coefficients of the equation of a straight line in such a form (which is called the intercept form of the equation of a straight line) have a simple geometrical meaning: *α and β are equal (up to a sign) to the lengths of the line segments intercepted by the straight line on the coordinate axes* (Fig. 18). Indeed, the straight line intersects both the x-axis ($y = 0$) at point (α, 0), and the y-axis ($x = 0$) at point (0, β).

EXERCISES

1. Under what condition does the straight line

$$ax + by + c = 0$$

intersect the positive semi-axis x (the negative semi-axis x)?

2. Under what condition does the straight line

$$ax + by + c = 0$$

not intersect the first quadrant?

3. Show that the straight lines given by the equations

$$ax + by + c = 0, \quad ax - by + c = 0, \quad b \neq 0,$$

are situated symmetrically about the x-axis.

4. Show that the straight lines specified by the equations

$$ax + by + c = 0; \quad ax + by - c = 0,$$

are arranged symmetrically about the origin.

5. Given a *pencil of lines*

$$ax + by + c + \lambda (a_1x + b_1y + c_1) = 0.$$

Find out for what value of the parameter λ is a line of the pencil parallel to the x-axis (y-axis); for what value of λ does the line pass through the origin?

6. Under what condition does the straight line

$$ax + by + c = 0$$

bound, together with the coordinate axes, an isosceles triangle?

7. Show that the area of the triangle bounded by the straight line

$$ax + by + c = 0 \quad (a, b, c \neq 0)$$

and the coordinate axes is

$$S = \frac{1}{2} \frac{c^2}{|ab|}.$$

8. Find the tangent lines to the circle

$$x^2 + y^2 + 2ax + 2by = 0,$$

which are parallel to the coordinate axes.

Sec. 2-3. The Equation of a Straight Line in the Form Solved with Respect to y. The Angle Between Two Straight Lines

When moving along any straight line not parallel to the y-axis x increases in one direction and decreases in the other. The direction in which x increases will be called *positive*.

Suppose we are given two straight lines g_1 and g_2 in the xy-plane which are not parallel to the y-axis. The angle

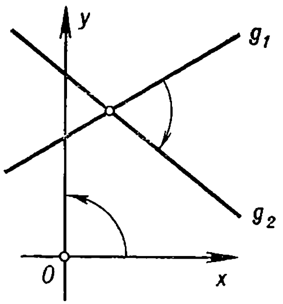


Fig. 19.

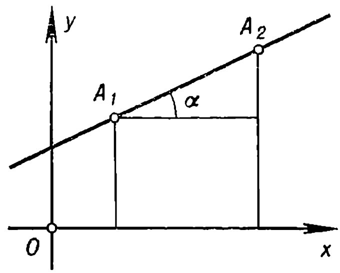


Fig. 20.

$\theta(g_1, g_2)$ formed by the line g_2 with the line g_1 is defined as an angle, less than π by absolute value, through which the line g_1 must be turned so that the positive direction on it is brought in coincidence with the positive direction on g_2. This angle is considered to be positive if the line g_1 is turned in the same direction in which the positive semi-axis x is turned through the angle $\pi/2$ until it coincides with the positive semi-axis y (Fig. 19).

The angle between the straight lines possesses the following obvious properties:

(1) $\theta(g_1, g_2) = -\theta(g_2, g_1)$;

(2) $\theta(g_1, g_2) = 0$ when and only when the lines are parallel or coincide;

(3) $\theta(g_3, g_1) = \theta(g_3, g_2) + \theta(g_2, g_1)$.

Let

$$ax + by + c = 0$$

be a straight line not parallel to the y-axis ($b \neq 0$). Multiplying the equation of this line by $1/b$ and putting $-a/b = k$, $-c/b = l$, we reduce it to the form

$$y = kx + l. \qquad (*)$$

The coefficients of the equation of a straight line in this form have a simple geometrical meaning:

k is the tangent of the angle α formed by the straight line with the x-axis;

l is the line segment (up to a sign) intercepted by the straight line on the y-axis.

Indeed, let A_1 (x_1, y_1) and A_2 (x_2, y_2) be two points on the straight line (Fig. 20). Then

$$\tan\alpha = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(kx_2 + l) - (kx_1 + l)}{x_2 - x_1} = k.$$

The y-axis ($x = 0$) is obviously intersected by the line at point $(0, l)$.

Let there be given in the xy-plane two straight lines:

$$y = k_1 x + l_1,$$
$$y = k_2 x + l_2.$$

Let us find the angle θ formed by the second line with the first one. Denoting by α_1 and α_2 the angles formed by the straight lines with the x-axis, by virtue of property (3) we get

$$\theta = \alpha_2 - \alpha_1.$$

Since the angular coefficients $k_1 = \tan\alpha_1$, $k_2 = \tan\alpha_2$, we get

$$\tan\theta = \frac{k_2 - k_1}{1 + k_1 k_2}.$$

Whence θ is determined, since $|\theta| < \pi$.

EXERCISES

1. Show that the straight lines $ax + by + c = 0$ and $bx - ay + c' = 0$ intersect at right angles.

2. What angle is formed with the x-axis by the straight line

$$y = x\cot\alpha, \quad \text{if} \quad -\frac{\pi}{2} < \alpha < 0?$$

3. Form the equations of the sides of a right-angled triangle whose side is equal to 1, taking one of the sides and the altitude for the coordinate axes.

4. Find the interior angles of the triangle bounded by the straight lines $x + 2y = 0$, $2x + y = 0$, and $x + y = 1$.

5. Under what condition for the straight lines $ax + by = 0$ and $a_1x + b_1y = 0$ is the x-axis the bisector of the angles formed by them?

6. Derive the formula $\tan\theta = \frac{c}{a}$ for the angle θ formed by the straight line $x = at + b$, $y = ct + d$ with the x-axis.

7. Find the angle between the straight lines represented by the equations in parametric form:

$$\left.\begin{aligned} x &= a_1t + b_1, \\ y &= a_2t + b_2; \end{aligned}\right\} \quad \left.\begin{aligned} x &= c_1t + d_1, \\ y &= c_2t + d_2. \end{aligned}\right\}$$

8. Show that the quadrilateral bounded by the straight lines

$$\pm ax \pm by + c = 0 \quad (a,\ b,\ c \neq 0),$$

is a rhombus and the coordinate axes are its diagonals.

Sec. 2-4. The Parallelism and Perpendicularity Conditions of Two Straight Lines

Suppose we have in the xy-plane two straight lines given by the equations

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0.$$

Let us find out what condition must be satisfied by the coefficients of the equations of the straight lines for these lines to be (a) parallel to each other, (b) mutually perpendicular.

Assume that neither of the straight lines is parallel to the y-axis. Then their equations may be written in the form

$$y = k_1x + l_1, \quad y = k_2x + l_2,$$

where

$$k_1 = -\frac{a_1}{b_1}, \quad k_2 = -\frac{a_2}{b_2}.$$

Taking into account the expression for the angle between straight lines, we get the *parallelism condition of two straight lines*:

$$k_1 - k_2 = 0,$$

or

$$a_1b_2 - a_2b_1 = 0. \qquad (*)$$

The perpendicularity condition of straight lines:

$$1 + k_1k_2 = 0,$$

or

$$a_1a_2 + b_1b_2 = 0. \qquad (*)$$

Though the conditions (*) and (**) are obtained in the assumption that neither of the straight lines is parallel to the y-axis, they remain true even if this condition is violated.

Let for instance, the first straight line be parallel to the y-axis. This means that $b_1 = 0$. If the second line is parallel to the first one, then it is also parallel to the y-axis, and, consequently, $b_2 = 0$. The condition (*) is obviously fulfilled. If the second line is perpendicular to the first one, then it is parallel to the x-axis and, consequently, $a_2 = 0$. In this case the condition (**) is obviously fulfilled.

Let us now show that *if the condition (*) is fulfilled for the straight lines, then they are either parallel, or coincide.*

Suppose, $b_1 \neq 0$. Then it follows from the condition (*) that $b_2 \neq 0$, since if $b_2 = 0$, then a_2 is also equal to zero which is impossible. In this event the condition (*) may be written in the following way

$$-\frac{a_1}{b_1} = -\frac{a_2}{b_2}, \quad \text{or} \quad k_1 = k_2,$$

which expresses the equality of the angles formed by the straight lines with the x-axis. Hence, the lines are either parallel, or coincide.

If $b_1 = 0$ (which means that $a_1 \neq 0$), then it follows from (*) that $b_2 = 0$. Thus, both straight lines are parallel to the y-axis and, consequently, they are either parallel to each other, or coincide.

Let us show that the *condition* (**) *is sufficient for the lines to be mutually perpendicular.*

Suppose $b_1 \neq 0$ and $b_2 \neq 0$. Then the condition (**) may be rewritten as follows:

$$1+\left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right)=0,$$

or

$$1 + k_1k_2 = 0.$$

This means that the straight lines form a right angle, i.e. they are mutually perpendicular.

If then $b_1 = 0$ (hence, $a_1 \neq 0$), we get from the condition (**) that $a_2 = 0$. Thus, the first line is parallel to the y-axis, and the second one is parallel to the x-axis which means that they are perpendicular to each other.

The case when $b_2 = 0$ is considered analogously.

EXERCISES

1. Show that two straight lines intercepting on the coordinate axes segments of equal lengths are either parallel, or perpendicular to each other.

2. Find the parallelism (perpendicularity) condition of the straight lines represented by the equations in parametric form:

$$\left.\begin{aligned} x &= \alpha_1 t + a_1, \\ y &= \beta_1 t + b_1, \end{aligned}\right\} \quad \left.\begin{aligned} x &= \alpha_2 t + a_2, \\ y &= \beta_2 t + b_2. \end{aligned}\right\}$$

3. Find the parallelism (perpendicularity) condition for two straight lines one of which is specified by the equation

$$ax + by + c = 0,$$

the other being represented parametrically:

$$x = \alpha t + \beta, \quad y = \gamma t + \delta.$$

4. In a family of straight lines given by the equations

$$a_1x + b_1y + c_1 + \lambda (a_2x + b_2y + c_2) = 0$$

(λ, parameter of the family) find the line parallel (perpendicular) to the straight line

$$ax + by + c = 0.$$

Sec. 2-5. The Mutual Positions of a Straight Line and a Point. The Equation of a Straight Line in the Normal Form

Suppose we have in the xy-plane a point A' (x', y') and a straight line g:

$$ax + by + c = 0.$$

If the point A' lies on the line g, then

$$ax' + by' + c = 0.$$

Let us find out *what geometrical meaning has the expression*

$$h(x', y') \equiv ax' + by' + c,$$

if the point A' is not on the straight line.

Let A' (x', y') and A'' (x'', y'') be two points not lying on the line g. The coordinates of any point of the segment $A'A''$ can be represented in the form

$$x = tx' + (1 - t)x'', \; y = ty' + (1 - t)y'', \, 0 \leqslant t \leqslant 1$$

(cf. Sec. 1-3). Thus, for any point A of the segment $A'A''$

$$h(x, y) = th(x', y') + (1 - t)\, h(x'', y'') = h(t).$$

If the points A' and A'' belong to one half-plane, then $h(t)$ does not vanish on the interval $[0, 1]$. Consequently, $h(0) = h(x'', y'')$ and $h(1) = h(x', y')$ are of the same sign. If A' and A'' belong to different half-planes, then $h(t)$ vanishes on the interval $[0, 1]$ and, being a linear function, attains at the end-points values of opposite signs, i.e. $h(x'', y'')$ and $h(x', y')$ have opposite signs.

Hence, the *expression*

$$ax' + by' + c$$

is positive for the points A' belonging to one of the half-planes defined by the straight line g, and is negative for the points of the other.

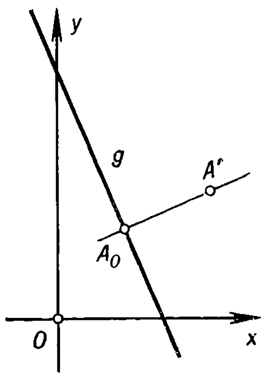


Fig. 21.

To find out geometrical meaning of $|ax' + by' + c|$ let us determine the distance of the point A' from the straight line g.

We drop from the point A' a perpendicular on the line g (Fig. 21). Let $A_0\ (x_0, y_0)$ be the foot of the perpendicular. The equation of the straight line $A'A_0$ can be written in the form

$$b(x - x') - a(y - y') = 0.$$

Indeed, the straight line represented by this equation passes through the point A' and is perpendicular to g. Hence

$$b(x_0 - x') - a(y_0 - y') = 0. \qquad (*)$$

Since the point A_0 lies on the line g we get,

$$ax_0 + by_0 + c = 0.$$

Whence

$$ax' + by' + c = a(x' - x_0) + b(y' - y_0). \quad (**)$$

Squaring (*) and (**) and adding them, we obtain

$$(ax' + by' + c)^2 = (a^2 + b^2)\,[(x'-x_0)^2 + (y'-y_0)^2].$$

Hence

$$|ax' + by' + c| = \sqrt{a^2 + b^2}\,\delta(x', y'),$$

where $\delta(x', y')$ is the distance of the point $A'(x', y')$ from the line g.

Thus, the *magnitude*

$$|ax' + by' + c|$$

is proportional to the distance of the point (x', y') *from the straight line*

$$ax + by + c = 0.$$

In particular, if $a^2 + b^2 = 1$, then this quantity is equal to the distance of the point from the straight line. In this case the straight line is said to be represented by an equation in the *normal form.*

As is obvious, to reduce the equation of the straight line

$$ax + by + c = 0$$

to the normal form it is sufficient to divide it by

$$+\sqrt{a^2+b^2} \quad \text{or} \quad -\sqrt{a^2+b^2}$$

As an example illustrating the application of the normal form of the equation of a straight line let us derive the formula for the area of a triangle given by the coordinates of its vertices. Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle. Then its area will be

$$S = \frac{1}{2} h\,|BC|,$$

where h is the altitude of the triangle dropped onto the side BC,

$$|BC| = [(x_2 - x_3)^2 + (y_2 - y_3)^2]^{1/2}.$$

We then find h. The equation of the straight line BC is

$$(x - x_2)(y_2 - y_3) - (y - y_2)(x_2 - x_3) = 0.$$

Indeed, it is linear and is satisfied by the points B and C. We now reduce this equation to the normal form by dividing it by $[(y_2 - y_3)^2 + (x_2 - x_3)^2]^{1/2}$. We get

$$\frac{(x-x_2)(y_2-y_3)-(y-y_2)(x_2-x_3)}{\sqrt{(x_2-x_3)^2+(y_2-y_3)^2}} = 0.$$

Substituting the coordinates of the vertex A in the left-hand member of this equation, we obtain (to within a digit) the altitude of the triangle dropped from the vertex A. Hence, the *area of the triangle*

$$S = \frac{1}{2} \mid (x_1 - x_2)(y_2 - y_3) - (y_1 - y_2)(x_2 - x_3) \mid .$$

EXERCISES

1. Given the equations of the sides of a triangle and a point by its coordinates. How can we find out whether this point lies inside the triangle or outside it?

2. Show that the distance between the parallel straight lines

$$ax + by + c_1 = 0, \quad ax + by + c_2 = 0$$

is equal to

$$\frac{\mid c_1 - c_2 \mid}{\sqrt{a^2 + b^2}}.$$

3. Form the equations of the straight lines parallel to the line

$$ax + by + c = 0,$$

and found at a distance δ from it.

4. Show that if two intersecting lines are represented by the equations in the normal form

$$ax + by + c = 0, \; a_1x + b_1y + c_1 = 0,$$

then the equations of the bisectors of the angles formed by them will be

$$(ax + by + c) \pm (a_1x + b_1y + c_1) = 0.$$

5. Show that the locus of points whose distances from two given straight lines are in a given ratio consists of two straight lines. Form the equations of these lines, taking the equations of the given lines in the normal form and putting the ratio of the distances to be equal to $\lambda : \mu$.

Sec. 2-6. Basic Problems on the Straight Line

Let us form the equation of an arbitrary straight line passing through the point A (x_1, y_1).

Suppose

$$ax + by + c = 0 \tag{*}$$

is the equation of the required line. Since the line passes through the point A, we get

$$ax_1 + by_1 + c = 0.$$

Expressing c and substituting it in the equation (*), we obtain

$$a(x - x_1) + b(y - y_1) = 0.$$

It is obvious that, for any a and b, the straight line given by this equation passes through the point A.

Let us form the equation of the straight line passing through two given points A_1 (x_1, y_1), A_2 (x_2, y_2).

Since the straight line passes through the point A_1, its equation may be written in the form

$$a(x - x_1) + b(y - y_1) = 0.$$

Since the line passes through the point A_2, we have

$$a(x_2 - x_1) + b(y_2 - y_1) = 0,$$

whence

$$\frac{a}{b} = -\frac{y_2 - y_1}{x_2 - x_1},$$

and the required equation will be

$$\frac{x - x_1}{x_2 - x_1} - \frac{y - y_1}{y_2 - y_1} = 0.$$

Let us now form the equation of a straight line parallel to the line

$$ax + by + c = 0,$$

and passing through the point A (x_1, y_1).

Whatever the value of λ, the equation

$$ax + by + \lambda = 0$$

represents a straight line parallel to the given one. Let us choose λ so that the equation is satisfied for $x = x_1$ and $y = y_1$:

$$ax_1 + by_1 + \lambda = 0.$$

Hence

$$\lambda = -ax_1 - by_1,$$

and the required equation will be

$$a(x - x_1) + b(y - y_1) = 0.$$

Let us form the equation of a straight line passing through the given point $A(x_1, y_1)$ and perpendicular to the line

$$ax + by + c = 0.$$

For any λ the straight line

$$bx - ay + \lambda = 0$$

is perpendicular to the given line. Choosing λ so that the equation is satisfied for $x = x_1$, $y = y_1$ we find the required equation

$$b(x - x_1) - a(y - y_1) = 0.$$

Let us form the equation of a straight line passing through the given point $A(x_1, y_1)$ at an angle α to the x-axis.

The equation of the straight line can be written in the form

$$y = kx + l.$$

The coefficients k and l are found from the conditions

$$\tan\alpha = k, \quad y_1 = kx_1 + l.$$

The required equation is

$$y - y_1 = (x - x_1)\tan\alpha.$$

We conclude with the following assertion: *the equation of any straight line passing through the point of intersection of two given straight lines*

$$a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0,$$

can be written in the form

$$\lambda(a_1x + b_1y + c_1) + \mu(a_2x + b_2y + c_2) = 0. \qquad (**)$$

Indeed, for any λ and μ which are not both zero, the equation (**) represents a straight line which passes through the point of intersection of the two given lines, since its coordinates obviously satisfy the equation (**). Further, whatever the point (x_1, y_1) which is different from the point of intersection of the given straight lines, the line (**) passes through the point (x_1, y_1) when

$$\lambda = a_1x_1 + b_2y_2 + c_2, \quad -\mu = a_1x_1 + b_1y_1 + c_1.$$

Consequently, the straight lines represented by (**) exhaust all the lines passing through the point of intersection of the given straight lines.

EXERCISES

1. Form the equation of a straight line parallel (perpendicular) to the straight line

$$ax + by + c = 0,$$

passing through the point of intersection of the straight lines

$$a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0.$$

2. Under what condition are the points (x_1, y_1), (x_2, y_2) situated symmetrically about the straight line

$$ax + by + c = 0?$$

3. Form the equation of a straight line passing through the point (x_0, y_0) and equidistant from the points (x_1, y_1) and (x_2, y_2).

4. Show that three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) lie on a straight line if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Sec. 2-7. Transformation of Coordinates

Let there be introduced two coordinate systems (xy and $x'y'$) in the plane (Fig. 22). *We have to establish the relation between the coordinates of an arbitrary point with respect to these coordinate systems.*

Let

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0$$

be the equations of the axes y' and x' in the normal form in the coordinate system xy.

The equation of a straight line in the normal form is defined uniquely to within a change of sign of all the coefficients of the equation. Therefore, without limitation of generality, we may assume that for some point A_0 (x_0, y_0) situated in the first quadrant of the coordinate system $x'y'$

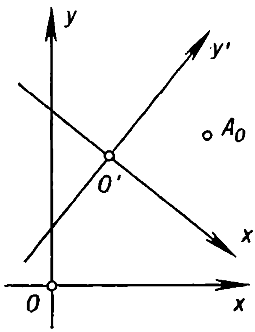


Fig. 22.

$$a_1x_0 + b_1y_0 + c_1 > 0,$$
$$a_2x_0 + b_2y_0 + c_2 > 0$$

(otherwise the signs of the coefficients may be reversed).

We assert that the *coordinates x', y' of an arbitrary point with reference to the coordinate system $x'y'$ are expressed in terms of the coordinates x, y of the same point in the coordinate system xy by the formulas*

$$\left.\begin{aligned} x' &= a_1x + b_1y + c_1, \\ y' &= a_2x + b_2y + c_2. \end{aligned}\right\} \qquad (*)$$

Let us, for instance, prove the first formula. The absolute value of its left-hand side is equal to the absolute value of its right-hand side, since it represents the distance of the point from the y'-axis. In each of the half-planes defined by the y'-axis both sides of the formula preserve the sign and change it when passing from one half-plane to the other. And since the signs coincide for the points A_0, they coincide for any point of the plane.

The second formula is proved in a similar way.

Since

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0$$

are the normal equations of two mutually perpendicular straight lines, the coefficients a_1, b_1, a_2, b_2, of the formulas (*) are related by the formulas

$$\left.\begin{aligned} a_1^2+b_1^2=1, \\ a_2^2+b_2^2=1, \\ a_1a_2+b_1b_2=0. \end{aligned}\right\} \qquad (**)$$

Taking into consideration the first two formulas of (**), we may represent the coefficients a_1, b_1, a_2, b_2 in the following way:

$$a_1 = \cos\alpha, \quad b_1 = \sin\alpha,$$
$$a_2 = \cos\alpha_1, \quad b_2 = \sin\alpha_1.$$

Then from the third relation of (**) we get

$$\cos\alpha\cos\alpha_1 + \sin\alpha\sin\alpha_1 = \cos(\alpha-\alpha_1) = 0,$$

whence it follows that $\alpha_1 = \alpha \pm \frac{\pi}{2} + 2k\pi$. Thus, the formulas (*) for the transformation of coordinates can be written in one of the following two forms:

$$\left.\begin{aligned} x' &= x\cos\alpha + y\sin\alpha + c_1, \\ y' &= -x\sin\alpha + y\cos\alpha + c_2 \end{aligned}\right\}$$

or

$$\left.\begin{aligned} x' &= x\cos\alpha + y\sin\alpha + c_1, \\ y' &= x\sin\alpha - y\cos\alpha + c_2. \end{aligned}\right\}$$

The first of them covers all the cases when the coordinate system $x'y'$ can be obtained from the coordinate system xy by motion. The second system of formulas suits for the cases when the coordinate system $x'y'$ is obtained from the system xy by means of motion and mirror reflection.

The quantities α, c_1, and c_2 in the formulas for transforming ordinates have a simple geometrical meaning: α is the angle formed by the x'-axis with the x-axis (to within an even 2π), and c_1 and c_2 are the coordinates of the origin of the coordinate system xy in the coordinate system $x'y'$.

Formulas for the transformation of coordinates allow another important interpretation if they are considered as formulas for onto mapping of the plane in which a point with the coordinates x', y' is correlated with the point with the coordinates xy *in the same coordinate system.* This mapping differs in that it preserves distances. Namely, the distance between any two points A and B is equal to the distance between their images A' and B'. Thus, *this mapping is a motion, or a motion with mirror reflection.* The first system of formulas corresponds to proper motion, whereas the second system of formulas gives motion with mirror reflection.

EXERCISES

1. Derive the formulas for passing from the coordinate system xy to the coordinate system $x'y'$ if the coordinate axes x' and y' are given by the equations

$$ax + by + c_1 = 0, \quad -bx + ay + c_2 = 0.$$

2. Derive the equation of the curve $x^2 - y^2 = a^2$, taking the straight lines

$$x + y = 0, \quad x - y = 0$$

for the new coordinate axes.

3. The new coordinate system $x'y'$ is obtained from the old coordinate system xy by rotating the latter about some point (x_0, y_0). Using the formulas (*), for the transformation of coordinates find x_0 and y_0.

4. Putting $z = x + iy$, show that any motion in the xy-plane is realized by a linear transformation of the complex variable

$$z' = \omega z + c,$$

where ω and c are complex numbers, and $|\omega| = 1$.

5. Find the equation of the curve described by the point C of the mechanism shown in Fig. 23. ABC is a rigid triangle, the point A slides along the x-axis, and the point B moves along a circle of radius R with centre at the origin.

Solution. At the moment when the point B coincides with B_0 the points A, B, and C have the coordinates $(d, 0)$, $(R, 0)$, and (a, b), respectively. Let us put $z_0 =$

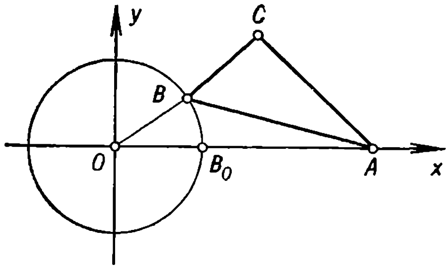


Fig. 23.

$= a + ib$. At an arbitrary moment the complex coordinate of the point C

$$z = \omega z_0 + c.$$

Since all the time the point B remains on the circle

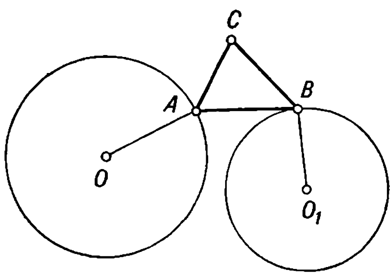


Fig. 24.

$x^2 + y^2 = R^2$, and the point A on the x-axis, we have

$$|\omega R + c| = R, \quad \operatorname{Im}(\omega d + c) = 0.$$

Hence

$$|\omega(R - z_0) + z| = R, \quad \operatorname{Im}(\omega(d - z_0) + z) = 0,$$

or

$$|R - z_0|^2 + \omega (R - z_0)\bar{z} + \bar{\omega}(R - \bar{z}_0) z + |z|^2 = R^2,$$
$$\omega (d - z_0) - \bar{\omega}(d - \bar{z}_0) + z - \bar{z} = 0$$

(conjugate complex numbers are marked with dashes).

Solving these equations with respect to ω and $\bar{\omega}$, and noting that $\omega\bar{\omega} = 1$, we find the equation satisfied by z. Substituting then $x + iy$ for z, we get the equation of the required curve.

6. Find the equation of the curve described by the point C of the mechanism shown in Fig. 24. The triangle ABC is rigid, its vertices A and B move along circles.

Chapter 3

Conic Sections

Sec. 3-1. Polar Coordinates

In a plane (Fig. 25) we take an arbitrary point O and draw a ray g. The direction of angular measurement about the point O is also given. Then the position of any point A in the plane may be specified by two numbers ρ and θ : (1) ρ expresses the distance of the point A from O, and (2) θ is the angle formed by the ray OA with the ray g.

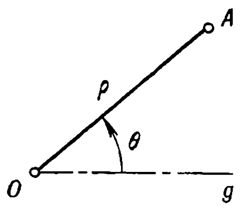


Fig. 25.

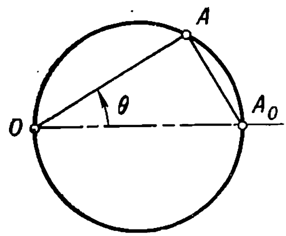


Fig. 26.

The numbers ρ and θ are called the *polar coordinates* of the point A. The point O is termed the *pole*, and the ray g the *polar axis.*

Like in the case of the Cartesian coordinates, we may speak of the equation of a curve in the polar coordinates. Namely, the equation

$$\varphi(\rho, \theta) = 0$$

is called the *equation of a curve in the polar coordinates* if it is satisfied by the polar coordinates of each point of the curve. And conversely, any pair of the numbers ρ, θ satisfying this equation represents the polar coordinates of one of the points on the curve.

By way of example let us form the equation (in polar coordinates) of a circle passing through the pole with centre on the polar axis and radius R. From a right-angled triangle OAA_0 we get $OA = OA_0 \cos\theta$ (Fig. 26). Whence the equation of the circle is

$$\rho = 2R \cos\theta.$$

Let us now introduce on the plane $\rho\theta$ a system of Cartesian coordinates xy, taking the pole O for the origin of the Cartesian coordinate system, the polar axis for the positive semi-axis x, and choosing the direction of the positive semi-axis y so that it forms an angle of $+\pi/2$ with the polar axis as measured in the chosen direction.

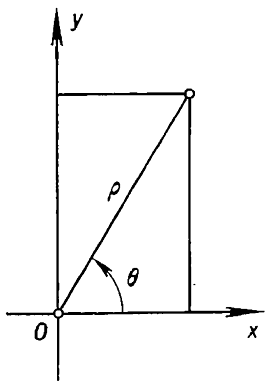


Fig. 27.

The following simple relationship is obviously established between polar and rectangular coordinates of a point:

$$x = \rho\cos\theta, \quad y = \rho\sin\theta \quad (*)$$

(Fig. 27). This makes it possible to get the equation of a curve in Cartesian coordinates, given the equation of this curve in polar coordinates, and vice versa.

Let us, for instance, form the equation of an arbitrary straight line in the polar coordinates. The equation of this line in the Cartesian coordinates is

$$ax + by + c = 0, \quad c < 0.$$

Introducing ρ and θ in this equation (instead of x and y) according to the formulas $(*)$, we get

$$\rho\,(a\cos\theta + b\sin\theta) + c = 0,$$

Putting then

$$\frac{a}{\sqrt{a^2+b^2}}=\cos\alpha,$$
$$\frac{b}{\sqrt{a^2+b^2}}=\sin\alpha,$$
$$\frac{c}{\sqrt{a^2+b^2}}=-\rho_0,$$

we obtain the equation of the straight line in the form

$$\rho\cos(\alpha-\theta)=\rho_0.$$

EXERCISES

1. Show that the equation of any circle in polar coordinates can be written in the form

$$\rho^2+2a\,\rho\cos(\alpha+\theta)+b=0.$$

Determine the coordinates of its centre ρ_0, θ_0, and the radius R.

2. Express the distance between two points in terms of the polar coordinates of these points.

3. What geometrical meaning have α and ρ_0 in the equation of a straight line in polar coordinates

$$\rho\cos(\alpha-\theta)=\rho_0?$$

4. Form the equation (in polar coordinates) of the locus of the feet of perpendiculars dropped from the point A on the circle onto its tangent lines (the *cardioid*, see Fig. 28). Take the point A for the pole, and the extension of the radius OA for the polar axis.

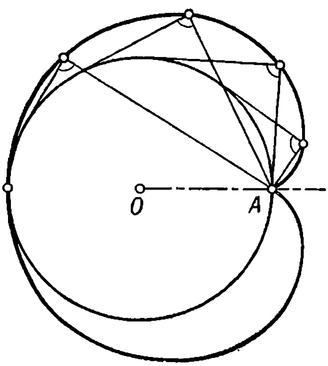


Fig. 28.

5. Form the equation of the *lemniscate of Bernoulli* which is the name for the locus of points the product of whose distances from two given points F_1 and F_2 (the foci) is constant and is equal

to $|F_1F_2|^2/4$. Take the mid-point of the line segment joining the foci for the pole, and the ray passing through one of the foci for the polar axis.

Sec. 3-2. Conic Sections and Their Equations in Polar Coordinates

A *conic section* (or a conic) is defined as a curve obtainable on the surface of a circular cone at the intersection with an arbitrary plane that does not pass through the

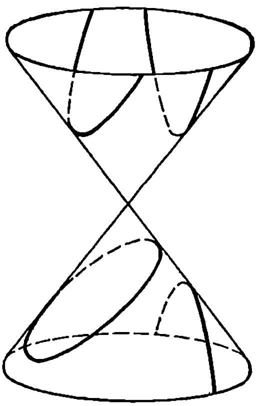


Fig. 29.

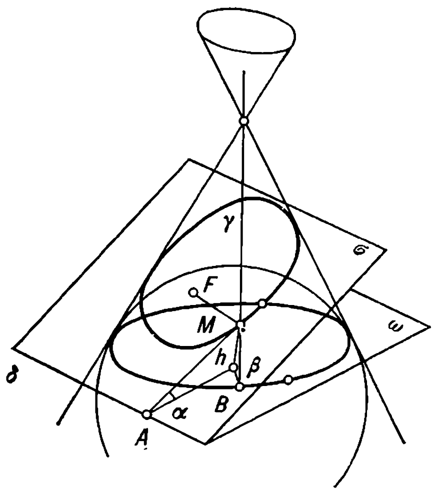


Fig. 30.

vertex of the cone (Fig. 29). Conics possess a number of remarkable properties, one of them consisting in the following.

Each conic section, except for a circle, is a plane locus of points the ratio of whose distances from a point F and a straight line δ is constant. The point F is called the *focus* of a conic, the straight line δ being its *directrix*.

Let us prove this property. Let γ be the curve along which the plane σ intersects the cone (Fig. 30). We now inscribe in the cone a sphere which touches the plane σ, and denote by F the point of contact of the sphere with the plane. Let ω denote the plane containing the circle along which the sphere touches the cone. We then take an arbitrary point M on the curve γ and draw through it an element of the cone, denoting by B the point at which it cuts the plane ω. We finally drop a perpendicular from the point M onto the line δ of intersection of the planes σ and ω.

It is stated that the curve γ possesses the above property with respect to the point F and the straight line δ. Indeed,

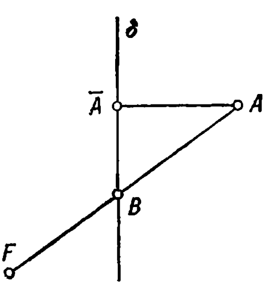


Fig. 31.

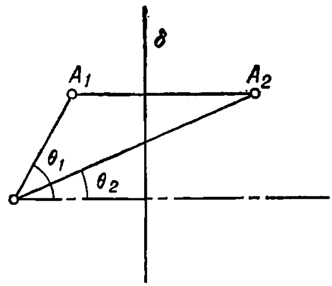


Fig. 32.

$FM = BM$ as tangent lines to the sphere drawn from one point. Further, if we denote by $h\ (M)$ the distance of the point M from the plane ω, then $AM = h\ (M)/\sin\alpha$, $BM = h\ (M)/\sin\beta$, where α is the angle between the planes ω and σ, and β is the angle between the generatrix of the cone and the plane ω.

Hence it follows that

$$\frac{AM}{FM} = \frac{AM}{BM} = \frac{\sin\beta}{\sin\alpha},$$

i.e. the ratio AM/FM does not depend on the point M. The statement has been proved.

Depending on the magnitude of the ratio λ of the distances of an arbitrary point of a conic section from the focus and directrix the curve is called the *ellipse* ($\lambda < 1$), the *parabola* ($\lambda = 1$), or the *hyperbola* ($\lambda > 1$), the number λ being termed the *eccentricity* of a conic section.

Let F be the focus of a conic section and δ its directrix (Fig. 31). In the case of the ellipse and parabola ($\lambda \leqslant 1$)

all points of the curve are situated on one side of the directrix, namely, on the side where the focus F is located. Indeed, for any point A situated on the other side of the directrix

$$\frac{AF}{A\bar{A}} > \frac{AB}{A\bar{A}} \geqslant 1.$$

On the contrary, the hyperbola ($\lambda > 1$) has points situated on both sides of the directrix. The hyperbola consists of two branches separated by the directrix.

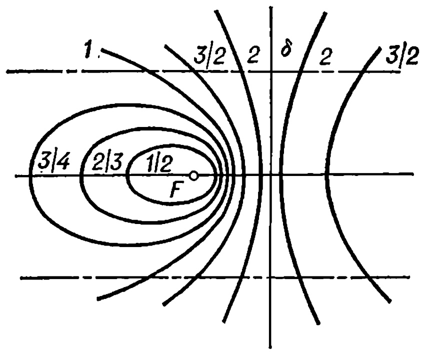


Fig. 33.

Let us form the equation of a conic section in polar coordinates taking the focus of the conic section for the pole of the coordinate system $\rho\theta$ and drawing the polar axis so that it is perpendicular to the directrix and intersects it (Fig. 32).

Suppose p is the distance of the focus from the directrix. The distance of an arbitrary point A of the conic section from the focus is equal to ρ, and the distance from the directrix to $p - \rho \cos \theta$ or $\rho \cos \theta - p$, depending on how the points A and F are situated relative to the directrix (on one or both of its sides). Hence the equation of the conic section

$$\frac{\rho}{p-\rho \cos \theta} = \lambda \qquad (*)$$

for the ellipse and parabola, and

$$\frac{\rho}{p-\rho\cos\theta}=\pm\lambda \qquad (**)$$

for the hyperbola (the upper sign corresponding to one branch of the hyperbola, and the lower sign to the other).

Solving the equations (*), (**) with respect to ρ, we get

$$\rho=\frac{\lambda p}{1+\lambda\cos\theta}$$

which is the *equation of the ellipse and parabola*, and

$$\rho=\frac{\pm\lambda p}{1\pm\lambda\cos\theta}$$

which is the *equation of the hyperbola.*

Figure 33 illustrates the change in the shape of a conic section depending on the eccentricity λ.

EXERCISES

1. Show that the curve

$$\rho=\frac{c}{1+a\cos\theta+b\sin\theta},$$

is a conic section. Under what condition is the curve an ellipse, a hyperbola, a parabola?

2. Given the three points $(\rho_1, 0)$, $(\rho_2, \pi/2)$, and (ρ_3, π), form the equation of an ellipse, knowing that one of its foci is situated at the pole of the $\rho\theta$ coordinate system.

3. Let A and B be the points at which a conic section intersects a straight line passing through the focus F. Prove that

$$\frac{1}{AF}+\frac{1}{BF}$$

does not depend on the straight line.

4. Show that the inversion of the parabola with respect to the focus transforms it into a cardioid (see Exercise 4 of Sec. 3-1).

Sec. 3-3. The Equations of Conic Sections in Rectangular Cartesian Coordinates in Canonical Form

In Sec. 3-2 we obtained the equations of conic sections in the polar coordinates $\rho\theta$. Let us now pass over to the rectangular coordinate system xy, taking the pole O for the origin and the polar axis for the positive semi-axis x.

From the equations (*) and (**) of Sec. 3-2 for any conic section we have

$$\rho^2 = \lambda^2 (p - \rho \cos \theta)^2.$$

Whence, taking into account the formulas of Sec. 3-1, which establish relation between the polar and Cartesian coordinates of a point, we obtain

$$x^2 + y^2 = \lambda^2 (p - x)^2,$$

or

$$(1 - \lambda^2) x^2 + 2p\lambda^2 x + y^2 - \lambda^2 p^2 = 0. \qquad (*)$$

This equation becomes considerably simplified, if we displace the origin along the x-axis in a required way.

Let us begin with the ellipse and hyperbola. In this case the equation (*) may be written in the following way:

$$(1-\lambda^2)\left(x + \frac{p\lambda^2}{1-\lambda^2}\right)^2 + y^2 - \frac{p^2\lambda^2}{1-\lambda^2} = 0.$$

We now introduce the new coordinates x', y', using the formulas

$$x + \frac{\lambda^2 p}{1-\lambda^2} = x', \quad y = y',$$

which corresponds to the transfer of the origin into the point

$$\left(-\frac{\lambda^2 p}{1-\lambda^2}, \ 0\right).$$

Then the equation of a curve will take the form

$$(1-\lambda^2) x'^2 + y'^2 - \frac{\lambda^2 p^2}{1-\lambda^2} = 0,$$

or, putting for brevity

$$\frac{\lambda^2 p^2}{(1-\lambda^2)^2}=a^2, \quad \frac{\lambda^2 p^2}{|1-\lambda^2|}=b^2,$$

we get the following equations:

for the ellipse

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1=0,$$

for the hyperbola

$$\frac{x'^2}{a^2}-\frac{y'^2}{b^2}-1=0.$$

The parameters a and b are termed the semi-axes of an ellipse (a hyperbola).

For the parabola ($\lambda = 1$) the equation (*) will have the form

$$2px + y^2 - p^2 = 0,$$

or

$$y^2-2p\left(-x+\frac{p}{2}\right)=0;$$

by introducing the new coordinates

$$x'=-x+\frac{p}{2}, \quad y'=y$$

it is transformed to the form

$$y'^2 - 2px' = 0.$$

The equations of the conic sections obtained in the coordinates x', y' are called *canonical.*

EXERCISES

1. Show that the equation of a conic section with the focus (x_0, y_0) and the directrix

$$ax + by + c = 0$$

has the form

$$(x - x_0)^2 + (y - y_0)^2 - k^2 (ax + by + c)^2 = 0.$$

For what values of k^2 is this conic section an ellipse, a parabola, a hyperbola?

2. Let K be any conic section and F its focus. Show that the distance of an arbitrary point A of the conic section from the focus F is linearly expressed in terms of the coordinates x, y of the point, i.e.

$$AF = \alpha x + \beta y + \gamma,$$

where α, β, γ are constants.

3. Show that any straight line intersects a conic section at most at two points.

4. Show that the locus of points the sum of whose distances from two given points is constant is an ellipse (see Exercise 4 of Sec. 1-4).

5. Show that the locus of points the difference of whose distances from two given points is constant is a hyperbola (see Exercise 5 of Sec. 1-4).

6. What is the locus of the centres of circles touching the two given circles K_1 and K_2? Consider various cases of mutual positions of the circles K_1 and K_2, and also the case when one of the circles degenerates into a straight line.

Sec. 3-4. Studying the Shape of Conic Sections

The ellipse (Fig. 34):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let us note here that the coordinate axes are the axes of symmetry of the ellipse, and the origin is the centre of symmetry. Indeed, if the point (x, y) belongs to the ellipse, then the points symmetrical to it about the coordinate axes $(-x, y)$, $(x, -y)$ and about the origin $(-x, -y)$ also belong to the ellipse, since they satisfy its equation together with the point (x, y). The points of intersection of the ellipse with its axes of symmetry are called the *vertices of the ellipse*.

The entire ellipse is contained inside a rectangle $|x| \leqslant a$, $|y| \leqslant b$ formed by the tangent lines to the ellipse at its vertices (see Fig. 35)

Indeed, if the point (x, y) is situated outside the rectangle, then at least one of the inequalities $|x| > a$ or

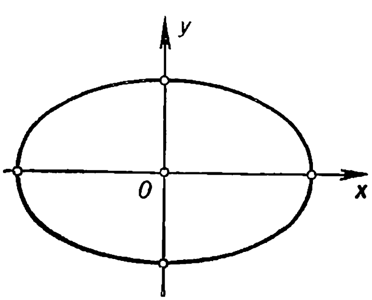


Fig. 34.

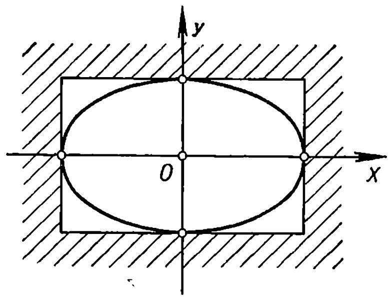


Fig. 35.

$|y| > b$ is satisfied for it, but then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}>1$$

and the point cannot belong to the ellipse.

We can obviously obtain an ellipse from a circle by uniformly contracting the latter. Let us draw on the plane a circle

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1. \qquad (*)$$

We then imagine that the xy-plane is uniformly contracted with respect to the x-axis so that the point (x, y) is transferred to the point $(\bar{x}, \bar{y})$, where $\bar{x} = x$, and $\bar{y} = \frac{b}{a}y$. In doing so the circle $(*)$ is transformed into a curve (Fig. 36). The coordinates of any of its points satisfy the equation

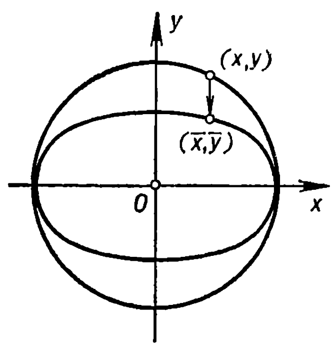


Fig. 36.

$$\frac{\bar{x}^2}{a^2}+\frac{\bar{y}^2}{b^2}=1.$$

Hence, this curve is an ellipse.

The hyperbola (Fig. 37):

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Just as in the case of the ellipse, we come to a conclusion that the coordinate axes are the axes of symmetry of the hyperbola, and the origin is the centre of symmetry.

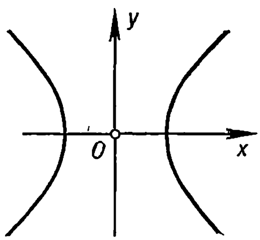


Fig. 37.

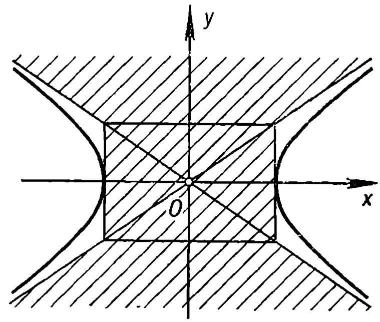


Fig. 38.

The hyperbola consists of two branches symmetrical about the y-axis and situated outside the rectangle $|x|<a$, $|y|<b$ and inside two angles formed by its extended diagonals (Fig. 38).

Indeed, inside the rectangle $|x|<a$ and, consequently,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}<1,$$

i.e. there are no points of the hyperbola inside the rectangle. Nor they exist within the hatched portion of the plane (see Fig. 38), since for any point (x, y) situated in this portion of the plane

$$\frac{b}{a}<\frac{|y|}{|x|},$$

whence

$$\frac{|x|}{a}<\frac{|y|}{b}.$$

and, consequently,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}<0<1.$$

It is worth mentioning the following property of the hyperbola. If a point (x, y), while moving off along the hyperbola, is at an infinite distance from the origin of the coordinates ($x^2+y^2 \to \infty$), then its distance from one of the diagonals of the rectangle which are obviously specified by the equations

$$\frac{x}{a}+\frac{y}{b}=0, \qquad \frac{x}{a}-\frac{y}{b}=0,$$

decreases infinitely (tends to zero).

Indeed, the quantities

$$\left|\frac{x}{a}+\frac{y}{b}\right| \quad \text{and} \quad \left|\frac{x}{a}-\frac{y}{b}\right|$$

are proportional to the distances of the point (x, y) of the hyperbola from the indicated lines (see Sec. 2-5). The product of these quantities

$$\left|\frac{x}{a}+\frac{y}{b}\right|\left|\frac{x}{a}-\frac{y}{b}\right|=\left|\frac{x^2}{a^2}-\frac{y^2}{b^2}\right|=1.$$

If our assertion that the distance from one of the diagonals tends to zero is false, then there exists such $\lambda>0$ and arbitrarily distant points of the hyperbola for which

$$\left|\frac{x}{a}+\frac{y}{b}\right|>\lambda, \quad \left|\frac{x}{a}-\frac{y}{b}\right|>\lambda.$$

And since

$$\left|\frac{x}{a}+\frac{y}{b}\right|\left|\frac{x}{a}-\frac{y}{b}\right|=1,$$

then for such points

$$\left|\frac{x}{a}+\frac{y}{b}\right|<\frac{1}{\lambda}, \quad \left|\frac{x}{a}-\frac{y}{b}\right|<\frac{1}{\lambda}.$$

Squaring these inequalities and adding them, we get

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}<\frac{1}{\lambda^2},$$

but this contradicts $x^2+y^2 \to \infty$.

The assertion has been proved.

The straight lines

$$\frac{x}{a}+\frac{y}{b}=0, \qquad \frac{x}{a}-\frac{y}{b}=0$$

are called the *asymptotes* of the hyperbola.

The hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$$

is said to be *conjugate* with respect to the considered hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

It has the same asymptotes but is situated inside the sup-

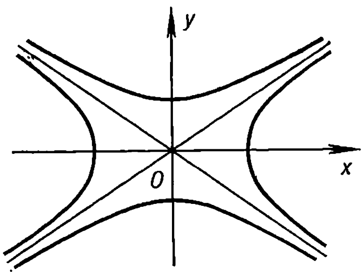


Fig. 39.

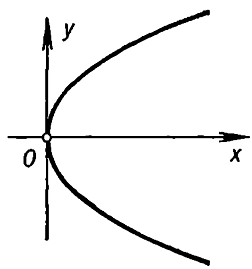


Fig. 40.

plementary vertical angles formed by the asymptotes (Fig. 39).

The parabola (Fig. 40):

$$y^2-2px=0$$

has the x-axis as the axis of symmetry, since along with the point (x, y) a point $(x, -y)$ which is symmetrical to it about the x-axis also belongs to the curve. The point of intersection of the parabola with its axis is called the *vertex* of the parabola. Thus, in this case the vertex of the parabola is the origin.

EXERCISES

1. Show that any ellipse is the projection of a circle.
2. Show that the product of the distances of a point on the hyperbola from its asymptotes is constant (i.e. it is independent of the point).

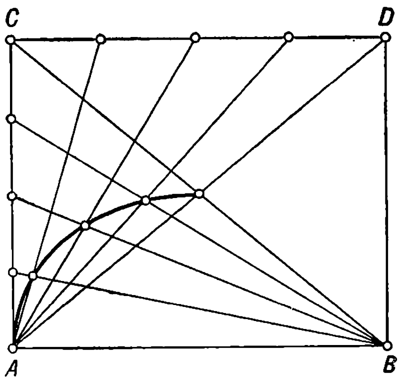


Fig. 41.

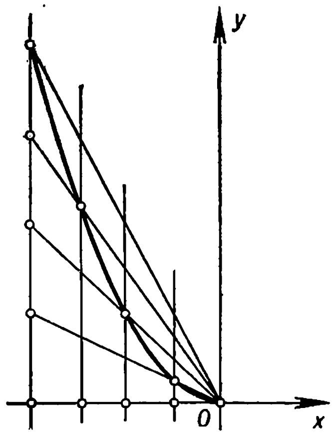


Fig. 42.

3. Show that the equation of any hyperbola with the asymptotes

$$a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0$$

can be written in the form

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = \text{const.}$$

4. Justify the following method of constructing an ellipse (Fig. 41). The sides of CD and AC of a triangle are divided into the same number of segments of equal length. The points of division are then joined to A and B. The points of intersection thus obtained lie on the ellipse with the major axis AB. The minor semi-axis is equal to half the altitude of the rectangle.
5. Justify the method of constructing the parabola illustrated in Fig. 42.

Sec. 3-5. A Tangent Line to a Conic Section

The *tangent line* to a curve at point A is defined as the limiting position, if this exists, of the secant line AB when the point B approaches A unboundedly (Fig. 43).

Suppose a curve is given by the equation $y = f(x)$. Let us form the equation of a tangent line at point $A(x_0, y_0)$. Let $B(x_0 + \Delta x, y_0 + \Delta y)$ be a point of the curve situated close to A. The equation of the secant is

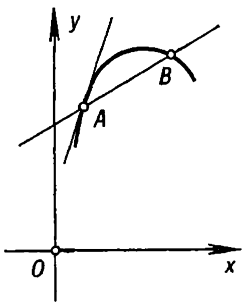


Fig. 43.

$$y - y_0 = \frac{\Delta y}{\Delta x}(x - x_0).$$

As $B \to A$

$$\frac{\Delta y}{\Delta x} \to f'(x_0),$$

and we get the *equation of the tangent line*

$$y - y_0 = f'(x_0)(x - x_0). \qquad (*)$$

Analogously, if a curve is specified by the equation $x = \varphi(y)$, then the equation of the tangent line at point (x_0, y_0) will be

$$x - x_0 = \varphi'(y_0)(y - y_0). \qquad (**)$$

Let us form the equation of a tangent line to a conic section.

The case of parabola. The equation of the parabola may be written in the form

$$x = \frac{y^2}{2p}.$$

Then the equation of the tangent line in the form $(**)$ will be

$$x - x_0 = \frac{y_0}{p}(y - y_0)$$

or

$$yy_0 - y_0^2 + px_0 - px = 0.$$

Since the point (x_0, y_0) lies on the parabola and, hence, $y_0^2 - 2px_0 = 0$, the equation of the tangent line can be represented in the following final form:

$$yy_0 - p(x + x_0) = 0.$$

The case of ellipse (hyperbola). Let (x_0, y_0) be a point on the ellipse, and $y_0 \neq 0$. In the neighbourhood of this point the ellipse can be specified by the equation

$$y = b\sqrt{1 - \frac{x^2}{a^2}},$$

where the square root should be taken with the same sign as y_0. The equation of the tangent line is found by the formula (*):

$$y - y_0 = -\frac{x_0 b}{a^2\sqrt{1 - \frac{x_0^2}{a^2}}}(x - x_0),$$

or

$$y - y_0 = -\frac{x_0 b^2}{y_0 a^2}(x - x_0).$$

Multiplying it by y_0/b^2 and transposing all terms to the left-hand side, we get

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}\right) = 0,$$

or

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - 1 = 0,$$

since $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$.

In the neighbourhood of any point (x_0, y_0) of the ellipse, where $x_0 \neq 0$ the ellipse can be specified by the equation

$$x = a\sqrt{1 - \frac{y^2}{b^2}}.$$

The square root is taken with the same sign as x_0. Then, reasoning in a similar way and using the formula (**) we arrive to the equation of the tangent line

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1.$$

Since at each point of the ellipse x_0 and y_0 cannot be equal both to zero, then at any point (x_0, y_0) the equation of the tangent line to the ellipse will be

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1.$$

The equation of the tangent line to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

is obtained analogously and has the form

$$\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1.$$

Let us show that a *tangent line to a conic section has only one point in common with this section* (i.e. the *point of tangency*). Indeed, let us take, for example, an ellipse whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The equation of the tangent line at point (x_0, y_0) will be

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1.$$

Let us now look for the points of intersection of the ellipse with its tangent line. Eliminating x from the equations, we obtain for y

$$\frac{y^2}{b^2}+\frac{a^2}{x_0^2}\left(\frac{yy_0}{b^2}-1\right)^2-1=0,$$

or

$$y^2\frac{a^2}{b^2x_0^2}\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}\right)-2y\,\frac{a^2}{x_0^2}\,\frac{y_0}{b^2}+\frac{a^2}{x_0^2}\left(1-\frac{x_0^2}{a^2}\right)=0.$$

Since the point (x_0, y_0) lies on the ellipse, we have $x_0^2/a^2 + y_0^2/b^2 = 1$, and the equation for y takes the form

$$\frac{a^2}{b^2x_0^2}(y^2-2yy_0+y_0^2)=0.$$

This equation has two merged roots $y = y_0$. Analogously, eliminating y from the equations of the ellipse and its tangent line we get $x = x_0$. Thus, the ellipse has only one point in common with the tangent line, i.e. the point of tangency (x_0, y_0). For the hyperbola and parabola this is proved in a similar way.

Based on the property of a tangent line to have only one point in common with a conic section is a refined method of deriving the equations of a pair of tangent lines passing through an arbitrary point. Let us take, for example, an ellipse specified by the following equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

We then form the equation of the tangent lines to the ellipse passing through the point (x_0, y_0) not lying on the ellipse. Let (x, y) be an arbitrary point. The coordinates of any point (x', y') on the straight line g passing through the points (x_0, y_0) and (x, y) can be represented in the form

$$x'=\frac{x_0+tx}{1+t},$$

$$y'=\frac{y_0+ty}{1+t}.$$

We now look for the points of intersection of the line g with the ellipse, for which purpose we substitute x' and y' in the equation of the ellipse. We get

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+2t\left(\frac{xx_0}{a^2}+\frac{yy_0}{b^2}\right)+t^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=(1+t)^2,$$

or

$$\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}-1\right)+2t\left(\frac{xx_0}{a^2}+\frac{yy_0}{b^2}-1\right)+$$
$$+t^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=0.$$

The point (x, y) will be on the tangent line to the ellipse if the roots of the equation for t are multiple, i.e. the discriminant of the equation is equal to zero. Hence, to get the equation of the tangent lines it is necessary to equate to zero the discriminant of the equation for t:

$$\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}-1\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)-\left(\frac{xx_0}{a^2}+\frac{yy_0}{b^2}-1\right)^2=0.$$

The equations of tangent lines to a hyperbola and parabola have an analogous form. Let us note here that the straight line

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$$

passes through the points of tangency.

EXERCISES

1. Show that a tangent line to the hyperbola, together with its asymptotes, defines a triangle of a constant area.

2. Express the condition of tangency of the straight line

$$y - y_0 = \lambda (x - x_0)$$

to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Show that the locus of the vertices (x_0, y_0) of right angles whose sides touch the ellipse is a circle.

3. Show that the vertices of right angles whose sides touch a parabola lie on the directrix, and the straight line joining the points of tangency passes through the focus.

4. Derive the equation of a pair of tangent lines to a conic section which are parallel to the straight line

$$\alpha x + \beta y + \gamma = 0.$$

5. Show that the segment of a tangent line to the hyperbola contained between the asymptotes is bisected by the point of tangency.

Sec. 3-6. The Focal Properties of Conic Sections

By definition, a conic section has a focus and a directrix. We are going to show that the *ellipse and hyperbola have one more focus and one more directrix.* Indeed, let the conic section be an ellipse. In the canonical arrangement its directrix δ_1 is parallel to the y-axis and the focus F_1 lies on the x-axis (Fig. 44). The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Since in such a position the ellipse is symmetrical with respect to the y-axis, it has a focus F_2 and a directrix δ_2 which are respectively symmetrical to the focus F_1 and the

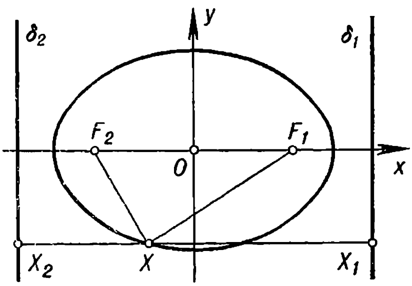


Fig. 44.

directrix δ_1 about the y-axis. Reasoning in an analogous way, we prove that the hyperbola also has two foci and two directrices.

Let us now show that the *sum of the distances of an arbitrary point of the ellipse from its foci is constant*, i.e. it is independent of the point. Indeed, for an arbitrary point X (Fig. 44) we have

$$\frac{XF_1}{XX_1}=\lambda, \quad \frac{XF_2}{XX_2}=\lambda.$$

Hence

$$XF_1+XF_2=\lambda\,(X_1X_2)=\text{const.}$$

Analogously, we can show that the *difference of the distances of an arbitrary point of the hyperbola from its foci is constant* (Fig. 45).

Let us find the foci of the ellipse and hyperbola in the canonical case.

The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Let c be the distance from the centre of the ellipse to the foci. The sum of the distances of the vertex $(0, b)$ from the foci is equal to $2\sqrt{b^2+c^2}$. The sum of the distances of the vertex $(a, 0)$ from the foci is equal to $2a$. Hence

$$\sqrt{b^2+c^2}=a,$$

and, consequently,

$$c=\sqrt{a^2-b^2}.$$

The equation of the hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

We then compare the difference between the distances of the point on the hyperbola with the abscissa c (where c is

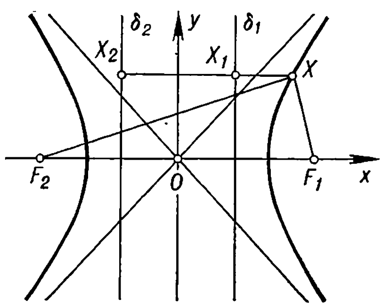


Fig. 45.

the distance from the centre of the hyperbola to the foci) with the difference between the distances of the vertex $(a, 0)$ from the foci. This comparison yields the following formula for the distance c

$$c=\sqrt{a^2+b^2}.$$

Let us mention the following optical or reflection property of the ellipse: *Rays of light emanating from one focus and being mirror reflected by the ellipse will come together at the other focus.* In other words, if A (x_0, y_0) is a point on the ellipse, then the segments AF_1 and AF_2 form equal angles with the tangent line at the point A.

To prove this property it is sufficient to show that the ratio of the distances of the focus from the tangent line and from the point of tangency A does not depend on what focus is taken: F_1 or F_2.

The square of the distance of the focus F_1 $(c, 0)$ from the point of tangency A (x_0, y_0) is

$$AF_1^2=(x_0-c)^2+y_0^2=(x_0-c)^2+\left(b^2-\frac{x_0^2b^2}{a^2}\right)=$$
$$=x_0^2\left(1-\frac{b^2}{a^2}\right)-2cx_0+b^2+c^2,$$

or, noting that $a^2=b^2+c^2$,

$$AF_1^2=\frac{x_0^2c^2}{a^2}-2cx_0+a^2=\left(\frac{cx_0}{a}-a\right)^2.$$

The distance of the focus F_1 $(c, 0)$ from the tangent line at the point A (x_0, y_0) is

$$h_1=k\left|\frac{cx_0}{a^2}-1\right|,$$

where k is a normalizing factor reducing the equation of the tangent line to the normal form.

Whence it follows that

$$\frac{h_1}{AF_1}=\frac{k}{a}.$$

For the other focus F_2 $(-c, 0)$ the same relation is obviously obtained. The assertion is thus proved.

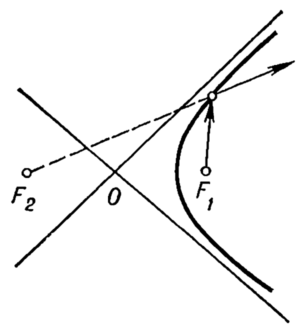


Fig. 46.

The hyperbola possesses a similar optical property: *Rays of light emanating from one focus seem to emanate from the other focus on being mirror reflected by the hyperbola* (Fig. 46).

The optical property of the parabola consists in that *rays of light emanating from its focus become parallel to its axis on being mirror reflected by the parabola.*

EXERCISES

1. Justify the following method of construction of the foci of the ellipse. From the vertex on the semiminor axis strike a circle of radius equal to the semimajor axis. Then the points of intersection of this circle with the major axis will be its foci.
2. Prove the optical property of the hyperbola.
3. Find the focus of the parabola in the canonical disposition.
4. Find the directrices of the conic sections in the canonical arrangement.
5. Show that all conic sections k_λ given by the equations

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

(where λ is the parameter of the family) are confocal, i.e. they have common foci.
6. Show that through any point of the xy-plane not belonging to the coordinate axes there pass two conic sections of the family k_λ (Exercise 5): an ellipse and a hyperbola.
7. Show that the ellipse and the hyperbola of the family k_λ (Exercise 5) passing through the point (x_0, y_0) intersect at this point at right angles, i.e. the tangent lines to them at the point (x_0, y_0) are mutually perpendicular.

Sec. 3-7. The Diameters of a Conic Section

The *diameter of an ellipse* (*a hyperbola*) is defined as any straight line passing through the centre of the ellipse (hyperbola). The *diameter of a parabola* is defined as any straight line parallel to its axis, in particular the axis itself.

An arbitrary straight line intersects a conic section at most at two points. If there are two points of intersection, then the line segment with the ends at the points of intersection is termed the *chord*. A conic section has the following property: *The mid-points of a set of its parallel chords lie on the diameter* (Fig. 47).

This property is obvious if the chords are perpendicular to the axis of symmetry. In this case the mid-points of the chords lie on this axis.

Consider the general case. A family of parallel straight lines not parallel to the coordinate axes can be specified by the following equations

$$y = kx + b, \quad k \neq 0,$$

where k is the same for all straight lines.

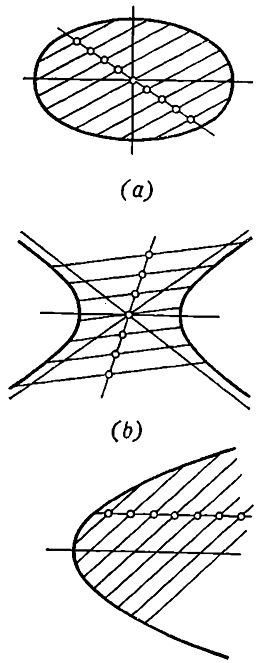
The equations of the ellipse and hyperbola can be combined in the following way:

$$\alpha x^2 + \beta y^2 - 1 = 0.$$

The end-points of the chords satisfy the system of equations

$$\alpha x^2 + \beta y^2 - 1 = 0,$$
$$y = kx + b.$$

Substituting $kx + b$ for y in the first equation, we find the equation which is satisfied by the abscissas x_1 and x_2 of the end-points of the chord:



(a) (b) (c)

Fig. 47.

$$(\alpha + \beta k^2) x^2 + 2\beta kbx + \beta b^2 - 1 = 0.$$

By the property of the roots of a quadratic equation

$$x_1 + x_2 = -\frac{2\beta kb}{\alpha + \beta k^2}.$$

Thus, the abscissa of the mid-point of the chord

$$x_c = \frac{x_1 + x_2}{2} = -\frac{\beta kb}{\alpha + \beta k^2}.$$

The ordinate y_c is found by substituting x_c in the equation of the chord $y = kx + b$:

$$y_c = -\frac{\beta k^2 b}{\alpha+\beta k^2} + b = \frac{\alpha b}{\alpha+\beta k^2}.$$

Whence

$$y_c = -\frac{\alpha}{\beta k} x_c.$$

Thus, the mid-points of the parallel chords $y = kx + b$ lie on the straight line passing through the origin, i.e. through the centre of the ellipse (hyperbola). Its slope

$$k' = -\frac{\alpha}{\beta k}.$$

The diameter

$$y = k'x$$

is called *conjugate* with respect to the diameter

$$y = kx,$$

parallel to the chords.

Obviously, conjugacy of the diameters is a mutual property, since the slope of the diameter conjugate to

$$y = k'x$$

is

$$-\frac{\alpha}{\beta k'} = k.$$

Let us consider the case of parabola. The coordinates of the end-points of the chords satisfy the system

$$y^2 - 2px = 0, \quad y = kx + b.$$

Eliminating x, we find the equation for the ordinates of the end-points:

$$y^2 - \frac{2py}{k} + \frac{2pb}{k} = 0.$$

Hence, like the previous case

$$y_1 + y_2 = \frac{2p}{k}.$$

Thus,

$$y_c = \frac{y_1 + y_2}{2} = \frac{p}{k} = \text{const.}$$

The mid-points of the chords lie on a straight line parallel to the x-axis (the axis of the parabola).

Let us mention one more property of the conjugate diameters: *If a diameter intersects a conic section, then the tangent lines at the points of intersection are parallel to the conjugate diameter.*

Indeed, let (x_0, y_0) be the point of intersection of the diameter $y = kx$ with the ellipse (hyperbola) $\alpha x^2 + \beta y^2 = 1$. The equation of the tangent line at the point (x_0, y_0) is $\alpha x x_0 + \beta y y_0 - 1 = 0$. Its slope $k' = -\alpha x_0/\beta y_0$. Since the point (x_0, y_0) lies on the diameter $y = kx$, we have $y_0 = kx_0$. Therefore

$$k' = -\frac{\alpha}{\beta k},$$

which was required to be proved.

EXERCISES

1. The tangent lines to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

have the slope k. Determine the points of tangency.

2. The chord of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is bisected at the point (x_0, y_0). Find the slope of the chord.

3. Show that the ellipse allows a parametric representation:

$$x = a \cos t, \quad y = b \sin t.$$

What condition is satisfied by the values of the parameter t corresponding to the end-points of the conjugate diameters? Prove that the sum of the squares of the lengths of the conjugate diameters of the ellipse is constant (Apollonius' theorem). Formulate and prove a similar theorem for the hyperbola.

4. Any ellipse can be represented as the projection of a circle. Show that in this projecting to the conjugate diameters of the ellipse there correspond the mutually perpendicular diameters of the circle. Relying on this fact, prove that the area of the parallelogram formed by the tangent lines at the end-points of the conjugate diameters is constant.

5. Show that the area of any parallelogram with the vertices at the end-points of the conjugate diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

has one and the same value equal to $2ab$.

6. It is known that of all the quadrilaterals inscribed in a circle the square has the greatest area. Show that among all the quadrilaterals inscribed in the ellipse the parallelograms with the vertices at the end-points of the conjugate diameters have the greatest area.

7. Show that the area of the ellipse with the semi-axes a and b is equal to πab.

8. Is it possible to inscribe a triangle in an ellipse so that the tangent line at each of its vertices is parallel to the opposite side? With what arbitrariness can it be done? What is the area of this triangle if the semi-axes of the ellipse are a and b.

Sec. 3-8. Second-Order Curves (Quadric Curves)

A curve of the second order is defined as the locus of points in the plane whose coordinates satisfy an equation of the form

$$a_{11}x^2+2a_{12}xy+a_{22}y^2+2a_1x+2a_2y+a=0, \quad (*)$$

in which at least one of the coefficients a_{11}, a_{12}, a_{22} is non-zero.

Obviously, *this definition is invariant relative to the choice of the coordinate system*, since the coordinates of a point in any other coordinate system are expressed linearly in terms of its coordinates in the xy-system and, consequently, the equation in any other coordinate system will have the form (*).

Let us find out the geometrical meaning of second-order curves. We put the curve in the new coordinate system $x'y'$ which is related to the xy-system by the formulas

$$x = x' \cos \alpha + y' \sin \alpha,$$
$$y = -x' \sin \alpha + y' \cos \alpha.$$

The equation of the curve, preserving the form (*), will have in the $x'y'$-system the coefficient

$$2a_1' = 2a_{11} \cos \alpha \sin \alpha - 2a_{22} \sin \alpha \cos \alpha + \\ + 2a_{12} (\cos^2 \alpha - \sin^2 \alpha) = \\ = (a_{11} - a_{22}) \sin 2\alpha + 2a_{12} \cos 2\alpha.$$

Obviously, it is always possible to choose the angle α so that this coefficient is equal to zero. Therefore, without limiting the generality, we may regard that in the initial equation (*) $a_{12} = 0$.

Further on we shall distinguish two cases:

Case A: both coefficients a_{11} and a_{22} are non-zero.

Case B: one of the coefficients a_{11} or a_{22} is equal to zero. Without limiting the generality, we shall consider $a_{11} = 0$.

In case A, by passing over to the new coordinate system $x'y'$,

$$x' = x + \frac{a_1}{a_{11}}, \quad y' = y + \frac{a_2}{a_{22}},$$

we bring the equation (*) to the form

$$a_{11}x'^2 + a_{22}y'^2 + c = 0 \qquad (**)$$

and consider the following subcases:

A$_1$: $c \neq 0$, a_{11} and a_{22} are of the same sign which is opposite to the sign of c. The curve is obviously an ellipse.

A$_2$: $c \neq 0$, a_{11} and a_{22} have different signs. The curve is a hyperbola.

A$_3$: $c \neq 0$, a_{11}, a_{22}, and c have the same sign. None of the real points satisfies the equation. The curve is called *imaginary*.

A_4: $c = 0$, a_{11} and a_{22} have different signs. The curve decomposes into two straight lines, since the equation (**) can be written in the form

$$\left(x' - \sqrt{-\frac{a_{22}}{a_{11}}}\,y'\right)\left(x' + \sqrt{-\frac{a_{22}}{a_{11}}}\,y'\right) = 0.$$

A_5: $c = 0$, a_{11} and a_{22} have the same sign. The equation can be written in the form

$$\left(x' - i\sqrt{\frac{a_{22}}{a_{11}}}\,y'\right)\left(x' + i\sqrt{\frac{a_{22}}{a_{11}}}\,y'\right) = 0.$$

The curve decomposes into a pair of imaginary straight lines intersecting at a real point (0, 0).

Let us now consider Case B.

In this case by passing over to the new coordinate system $x'y'$:

$$x' = x, \quad y' = y + \frac{a_2}{a_{22}},$$

the equation is reduced to the form

$$2a_1x' + a_{22}y'^2 + c = 0. \tag{***}$$

We then distinguish the following subcases:

B_1: $a_1 \neq 0$. The curve is a parabola, since by passing over to the new coordinates

$$x'' = x' + \frac{c}{2a_1}, \quad y'' = y',$$

the equatoin (***) is reduced to the form

$$2a_1x'' + a_{22}y''^2 = 0.$$

B_2: $a_1 = 0$, a_{22} and c have different signs. The curve decomposes into a pair of parallel straight lines

$$y \pm \sqrt{-\frac{c}{a_{22}}} = 0.$$

B_3: $a_1 = 0$, a_{22} and c are of the same sign. The curve decomposes into a pair of imaginary non-intersecting straight lines

$$y \pm i\sqrt{\frac{c}{a_{22}}} = 0.$$

B_4: $a_1 = 0$, $c = 0$. The curve is a pair of coinciding straight lines.

Thus, *a real curve of the second order represents either a conic section (the ellipse, hyperbola, parabola), or a pair of straight lines (which may even coincide).*

EXERCISES

1. Show that the second-order curve

$$(ax + by + c)^2 - (a_1x + b_1y + c_1)^2 = 0$$

decomposes into a pair of straight lines, may be coinciding ones.

2. As is known, all points of the ellipse are situated within a bounded portion of the xy-plane. Proceeding from this fact, show that the second-order curve $(ax + by + c)^2 + (\alpha x + \beta y + \gamma)^2 = k^2$ is an ellipse if the expressions $ax + by$ and $\alpha x + \beta y$ are independent and $k > 0$.

3. Show that the second-order curve

$$(ax + by + c)(\alpha x + \beta y + \gamma) = k \neq 0$$

is a hyperbola, provided the expressions $ax + by$, $\alpha x + \beta y$ are independent.

4. Show that the second-order curve

$$(ax + by + c)^2 - (\alpha x + \beta y + \gamma)^2 = k \neq 0,$$

is a hyperbola if $ax + by$, $\alpha x + \beta y$ are independent.

5. Show that if a straight line intersects a second-order curve at three points, then the curve decomposes into a pair of straight lines may be coinciding ones.

6. Show that if two indecomposable curves of the second order have five points in common, then they coincide.

7. A curve is termed a third-order curve if it is specified by the equation $\varphi_3(x, y) = 0$, where $\varphi_3(x, y)$ is a polynomial of the third degree with respect to x and y. Show that if a third-order curve γ_3 has seven points in common with an indecomposable second-order γ_2, then it decomposes into the curve γ_2 and a straight line.

8. Let γ be a second-order curve, $A_1, \ldots, A_6$ the vertices of a hexagon inscribed in it, $\alpha_{ij}(x, y) = 0$ the equations of the sides joining A_i and A_j (Fig. 48). Show that the third-order curve $\alpha_{24}\alpha_{16}\alpha_{35} - \lambda\alpha_{34}\alpha_{26}\alpha_{15} = 0$

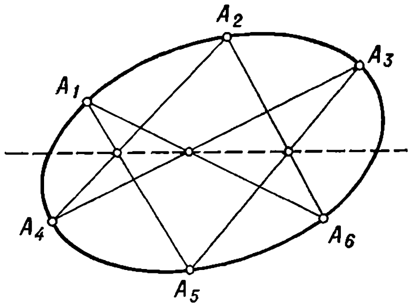


Fig. 48.

intersects the curve γ at six points A_i. Show that by a suitable choice of the parameter λ we can obtain a third-order curve which decomposes into the curve γ and a straight line.

9. Prove *Pascal's theorem*: The three points of intersection of the pairs of straight lines α_{15} and α_{24}, α_{34} and α_{16}, α_{26} and α_{35} lie on one line (Fig. 48).

Chapter 4

Vectors

Sec. 4-1. Addition and Subtraction of Vectors

In geometry, a *vector* is understood as a directed line segment (Fig. 49). The direction of a vector is indicated by the arrow. A vector with *initial point A* and *terminal point B* is denoted as $\overrightarrow{AB}$. A vector can also be denoted

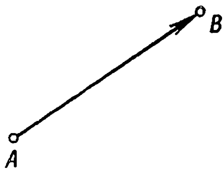


Fig. 49.

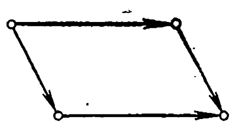


Fig. 50.

by a single letter. In printing this letter is given in boldface type (***a***), in writing it is given with a bar ($\overline{a}$).

Two vectors are considered to be *equal* if one of them can be obtained from the other by translation (Fig. 50). Obviously, if the vector ***a*** is equal to ***b***, then ***b*** is equal to ***a***. If ***a*** is equal to ***b***, and ***b*** is equal to ***c***, then ***a*** is equal to ***c***.

The vectors are said to be *in the same direction* (*in opposite directions*) if they are parallel, and the terminal points of two vectors equal to them and reduced to a common origin are found on one side of the origin (on different sides of the origin).

The length of the line segment depicting a vector is called the *absolute value* of the vector.

A vector of zero length (i.e. whose initial point coincides with the terminus) is termed the *zero vector*.

Vectors may be added or subtracted geometrically, i.e. we may speak of *addition* and *subtraction* of vectors. Namely, the *sum* of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is a third vector $\boldsymbol{a} + \boldsymbol{b}$ which is obtained from the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (or vectors equal to them) in the way shown in Fig. 51.

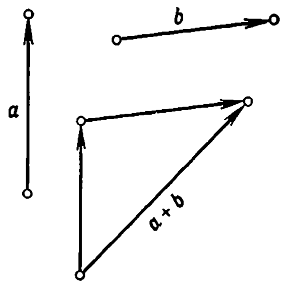


Fig. 51.

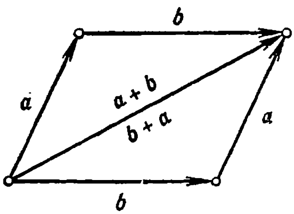


Fig. 52.

Vector addition is commutative, i.e. for any vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (Fig. 52).

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$$

Vector addition is associative, i.e. if $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are any vectors then

$$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}).$$

This property of addition, as also the preceding one, follows directly from the definition of the operation of addition (Fig. 53).

Let us mention here that if the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then the vector $\boldsymbol{a} + \boldsymbol{b}$ (if it is not equal to zero) is parallel to the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and is in the same direction with the greater (by absolute value) vector. The absolute value of the vector $\boldsymbol{a} + \boldsymbol{b}$ is equal to the sum of the absolute values of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ if they are in the same direction, and to the difference of the absolute values if the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are in opposite directions.

Subtraction of vectors is defined as the inverse operation of addition. Namely, the *difference* of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as the vector $\boldsymbol{a} - \boldsymbol{b}$ which, together with the vector $\boldsymbol{b}$, yields the vector $\boldsymbol{a}$. Geometrically it is obtained from the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (or vectors equal to them) as is shown in Fig. 54.

For any vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ we have the following inequality

$$|\boldsymbol{a}+\boldsymbol{b}| \leqslant |\boldsymbol{a}| + |\boldsymbol{b}|$$

(*the triangle inequality*), geometrically expressing the fact that in a triangle the sum of its two sides is greater than

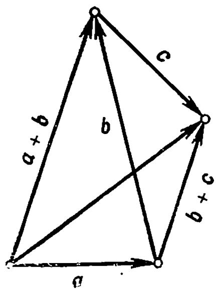


Fig. 53.

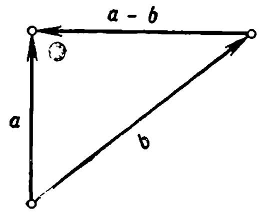


Fig. 54.

the third side if the vectors are not parallel. This inequality is obviously valid for any number of vectors:

$$|\boldsymbol{a}+\boldsymbol{b}+\ldots+\boldsymbol{l}| \leqslant |\boldsymbol{a}| + |\boldsymbol{b}| + \ldots + |\boldsymbol{l}|.$$

EXERCISES

1. Show that the sum of n vectors reduced to a common origin at the centre of a regular n-gon and with the terminal points at its vertices is equal to zero.

2. Three vectors have a common origin O and their terminal points are at the vertices of the triangle ABC. Show that

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 0$$

if and only if O is the point of intersection of the medians of the triangle.

3. Prove the identity

$$2|\boldsymbol{a}|^2 + 2|\boldsymbol{b}|^2 = |\boldsymbol{a}+\boldsymbol{b}|^2 + |\boldsymbol{a}-\boldsymbol{b}|^2.$$

To what geometrical fact does it correspond if a and b are non-zero and non-parallel vectors?

4. Show that the sign of equality in the triangle inequality takes place only when both vectors are in the same direction, or at least one of the vectors is equal to zero.

5. If the sum of the vectors $r_1, \ldots, r_n$ reduced to a common origin O is equal to zero and these vectors are not coplanar, then whatever is the plane α passing through the point O there can be found vectors r_i situated on both sides of the plane. Show this.

6. The vector r_{mn} lies in the xy-plane; its initial point is (x_0, y_0) and the terminus is the point $(m\delta, n\delta)$, where m and n are whole numbers not exceeding M and N by absolute value, respectively. Find the sum of all the vectors r_{mn} expressing it in terms of the vector with the initial point at (0, 0) and the terminus at the point (x_0, y_0).

7. A finite figure F in the xy-plane has the origin as the centre of symmetry. Show that the sum of the vectors with a common origin and termini at the points whose coordinates are whole numbers of the figure F is equal to zero if and only if the origin of coordinates serves as their common initial point. (It is assumed that the figure F has at least one point whose coordinates are whole numbers.)

8. Express the vectors represented by the diagonals of a parallelepiped in terms of the vectors represented by its edges.

Sec. 4-2. Multiplication of a Vector by a Number

Vectors may also be multiplied by a number. The *product* of the vector a by the number λ is defined as the vector $a\lambda = \lambda a$ the absolute value of which is obtained by multiplying the absolute value of the vector a by the absolute value of the number λ, i.e. $|\lambda a| = |\lambda|\,|a|$, the direction coinciding with the direction of the vector a or being in the opposite sense depending on whether $\lambda > 0$ or $\lambda < 0$. If $\lambda = 0$ or $a = 0$, then λa is considered to be equal to the zero vector.

The multiplication of a vector by a number possesses the associative property and two distributive properties. Namely, for any number λ, μ *and vectors* $\boldsymbol{a}$, $\boldsymbol{b}$

$$\lambda(\mu\boldsymbol{a}) = (\lambda\mu)\boldsymbol{a} \qquad \textit{(associative property)}$$

$$\left.\begin{aligned}(\lambda+\mu)\boldsymbol{a} &= \lambda\boldsymbol{a}+\mu\boldsymbol{a},\\ \lambda(\boldsymbol{a}+\boldsymbol{b}) &= \lambda\boldsymbol{a}+\lambda\boldsymbol{b}\end{aligned}\right\} \textit{(distributive properties)}$$

Let us prove these properties.

The absolute values of the vectors $\lambda(\mu\boldsymbol{a})$ and $(\lambda\mu)\boldsymbol{a}$ are the same and are equal to $|\lambda||\mu||\boldsymbol{a}|$. The directions of these vectors either coincide, if λ and μ are of the same sign, or are opposite if λ and μ have different signs. Hence, the vectors $\lambda(\mu\boldsymbol{a})$ and $(\lambda\mu)\boldsymbol{a}$ are equal by absolute value and are in the same direction, consequently, they are equal. If at least one of tbe numbers λ, μ or the vector $\boldsymbol{a}$ is equal to zero, then both vectors are equal to zero and, hence, they are equal to each other. The associative property is thus proved.

We are now going to prove the first distributive property:

$$(\lambda + \mu)\boldsymbol{a} = \lambda\boldsymbol{a} + \mu\boldsymbol{a}.$$

The equality is obvious if at least one of the numbers λ, μ or the vector $\boldsymbol{a}$ is equal to zero. Therefore, we may consider that λ, μ, and $\boldsymbol{a}$ are non-zero.

If λ and μ are of the same sign, then the vectors $\lambda\boldsymbol{a}$ and $\mu\boldsymbol{a}$ are in the same direction. Therefore, the absolute value of the vector $\lambda\boldsymbol{a} + \mu\boldsymbol{a}$ is equal to $|\lambda\boldsymbol{a}| + |\mu\boldsymbol{a}| = |\lambda||\boldsymbol{a}| + |\mu||\boldsymbol{a}| = (|\lambda| + |\mu|)|\boldsymbol{a}|$. The absolute value of the vector $(\lambda + \mu)\boldsymbol{a}$ is equal to $|\lambda + \mu||\boldsymbol{a}| = (|\lambda| + |\mu|)|\boldsymbol{a}|$. Thus, the absolute values of the vectors $(\lambda + \mu)\boldsymbol{a}$ and $\lambda\boldsymbol{a} + \mu\boldsymbol{a}$ are equal and they are in the same direction. Namely, for $\lambda > 0$, $\mu > 0$ their directions coincide with the direction of $\boldsymbol{a}$, and if $\lambda < 0$, $\mu < 0$ they are opposite to $\boldsymbol{a}$. The case when λ and μ have different signs is considered in a similar way.

Let us prove the second distributive property:

$$\lambda(\boldsymbol{a} + \boldsymbol{b}) = \lambda\boldsymbol{a} + \lambda\boldsymbol{b}.$$

The property is obvious if one of the vectors or the number λ is equal to zero. If the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\boldsymbol{b}$ can be represented in the form $\boldsymbol{b} = \mu\boldsymbol{a}$. And the second distributive property follows from the first one. Indeed,

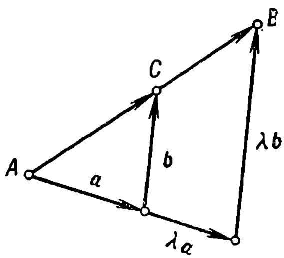


Fig. 55.

$$\lambda (1 + \mu)\, \boldsymbol{a} =$$

$$= \lambda (\boldsymbol{a} + \mu\boldsymbol{a}) = \lambda\boldsymbol{a} + \lambda\mu\boldsymbol{a}.$$

Hence,

$$\lambda (\boldsymbol{a} + \boldsymbol{b}) = \lambda\boldsymbol{a} + \lambda\boldsymbol{b}.$$

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be non-parallel vectors, then for $\lambda > 0$ the vector $\overrightarrow{AB}$ (Fig. 55) represents, on the one hand, $\lambda\boldsymbol{a} + \lambda\boldsymbol{b}$, and $\lambda\overrightarrow{AC}$ equal to $\lambda (\boldsymbol{a} + \boldsymbol{b})$ on the other. If $\lambda < 0$, then both vectors reverse their directions.

EXERCISES

1. The vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots$ are called *linearly independent* if there exist no numbers $\lambda_1, \lambda_2, \ldots$, (at least one of which is non-zero) such that

$$\lambda_1\boldsymbol{r}_1 + \lambda_2\boldsymbol{r}_2 + \ldots = 0$$

Show that two vectors are linearly independent if and only if they are non-zero and non-parallel.

Show that three vectors are linearly independent when and only when they are non-zero and there is no plane parallel to them.

2. Show that any three vectors lying in one plane are always linearly dependent.

3. Show that if two vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in a plane are linearly independent, then any vector $\boldsymbol{r}$ in this plane is expressed linearly in terms of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$

$$\boldsymbol{r} = \lambda_1\boldsymbol{r}_1 + \lambda_2\boldsymbol{r}_2.$$

The numbers λ_1 and λ_2 are defined uniquely.

4. Show that if three vectors r_1, r_2, r_3 are linearly independent, then any vector r is uniquely expressed in terms of these vectors in the form

$$r = \lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3.$$

Sec. 4-3. Scalar Product of Vectors

The *angle* between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as the angle between the vectors equal to $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, reduced to a common origin (Fig. 56).

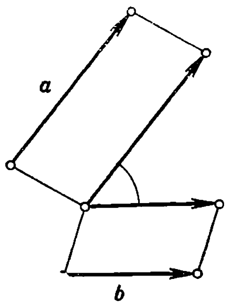


Fig. 56.

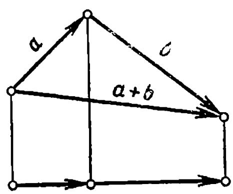


Fig. 57.

The *scalar product* of a vector $\boldsymbol{a}$ by a vector $\boldsymbol{b}$ is defined as the number $\boldsymbol{ab}$ which is equal to the product of the absolute value of the vectors by the cosine of the angle between them.

The scalar product possesses the following obvious properties which follow directly from its definition:

(1) $\boldsymbol{ab} = \boldsymbol{ba}$;

(2) $\boldsymbol{a}^2 = \boldsymbol{aa} = |\boldsymbol{a}|^2$;

(3) $(\lambda \boldsymbol{a})\,\boldsymbol{b} = \lambda\,(\boldsymbol{ab})$;

(4) *if* $|e| = 1$, *then* $(\lambda e)(\mu e) = \lambda\mu$;

(5) *the scalar product of vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *is equal to zero if and only if the vectors are mutually perpendicular or one of them is equal to zero.*

The *projection* of a vector $\boldsymbol{a}$ on a straight line is defined as the vector $\bar{\boldsymbol{a}}$ whose initial point is the projection of the initial point of the vector $\boldsymbol{a}$ and whose terminal point is the projection of the terminal point of the vector $\boldsymbol{a}$. Obviously, *equal vectors have equal projections, the projection of the sum of vectors is equal to the sum of the projections* (Fig. 57).

The scalar product of a vector $\boldsymbol{a}$ by a vector $\boldsymbol{b}$ is equal to the scalar product of the projection of the vector $\boldsymbol{a}$ onto the straight line containing the vector $\boldsymbol{b}$ by the vector $\boldsymbol{b}$. The proof is obvious. It is sufficient to note that $\boldsymbol{ab}$ and $\bar{\boldsymbol{a}}\boldsymbol{b}$ are equal by absolute value and have the same sign.

The scalar product possesses the distributive property. Namely for any three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$

$$(\boldsymbol{a}+\boldsymbol{b})\,\boldsymbol{c} = \boldsymbol{ac} + \boldsymbol{bc}.$$

The statement is obvious if one of the vectors is equal to zero. Let all the vectors be non-zero. Denoting by $\bar{\boldsymbol{a}}$, $\bar{\boldsymbol{b}}$, $\overline{\boldsymbol{a}+\boldsymbol{b}}$ the projections of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{a}+\boldsymbol{b}$ onto the line containing the vector $\boldsymbol{c}$, we have

$$(\boldsymbol{a}+\boldsymbol{b})\,\boldsymbol{c} = \overline{(\boldsymbol{a}+\boldsymbol{b})}\,\boldsymbol{c} = (\bar{\boldsymbol{a}}+\bar{\boldsymbol{b}})\,\boldsymbol{c},$$

$$\boldsymbol{ac}+\boldsymbol{bc} = \bar{\boldsymbol{a}}\boldsymbol{c} + \bar{\boldsymbol{b}}\boldsymbol{c}.$$

Let $\boldsymbol{e}$ be a unit vector parallel to $\boldsymbol{c}$. Then $\bar{\boldsymbol{a}}$, $\bar{\boldsymbol{b}}$, and $\boldsymbol{c}$ allow the representations $\bar{\boldsymbol{a}} = \lambda \boldsymbol{e}$, $\bar{\boldsymbol{b}} = \mu \boldsymbol{e}$, $\boldsymbol{c} = \nu \boldsymbol{e}$. We obtain

$$(\bar{\boldsymbol{a}}+\bar{\boldsymbol{b}})\,\boldsymbol{c} = (\lambda\boldsymbol{e}+\mu\boldsymbol{e})\,\nu\boldsymbol{e} = (\lambda+\mu)\,\nu,$$

$$\bar{\boldsymbol{a}}\boldsymbol{c}+\bar{\boldsymbol{b}}\boldsymbol{c} = \lambda\boldsymbol{e}\nu\boldsymbol{e}+\mu\boldsymbol{e}\nu\boldsymbol{e} = \lambda\nu+\mu\nu.$$

Whence

$$(\bar{\boldsymbol{a}}+\bar{\boldsymbol{b}})\,\boldsymbol{c} = \bar{\boldsymbol{a}}\boldsymbol{c}+\bar{\boldsymbol{b}}\boldsymbol{c}$$

and, hence,

$$(\boldsymbol{a}+\boldsymbol{b})\,\boldsymbol{c} = \boldsymbol{ac}+\boldsymbol{bc}.$$

In conclusion we are going to show that *if* $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ *are non-zero vectors which are not parallel to one plane, then from the three equalities*

$$\boldsymbol{ra} = 0, \quad \boldsymbol{rb} = 0, \quad \boldsymbol{rc} = 0$$

it follows that $\boldsymbol{r} = \boldsymbol{0}$.

Indeed, if $r \neq 0$, then from the above three equalities it follows that the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are perpendicular to $\boldsymbol{r}$, and therefore parallel to the plane perpendicular to $\boldsymbol{r}$ which is impossible.

EXERCISES

1. Let $A_1, A_2, \ldots, A_n$ be the vertices of a regular n-gon. Then $\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \ldots + \overrightarrow{A_nA_1} = 0$. Derive from this that

$$1 + \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \ldots + \cos\frac{(2n-2)\pi}{n} = 0,$$
$$\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \ldots + \sin\frac{(2n-2)\pi}{n} = 0.$$

2. Show that if $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero and non-parallel vectors, then $\lambda^2\boldsymbol{a}^2 + 2\mu\lambda\,(\boldsymbol{ab}) + \mu^2\boldsymbol{b}^2 \geqslant 0$, the equality to zero taking place only if $\lambda = 0$, and $\mu = 0$.

3. Show that for any three vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$ parallel to one and the same plane

$$\begin{vmatrix} \boldsymbol{r}_1\boldsymbol{r}_1 & \boldsymbol{r}_1\boldsymbol{r}_2 & \boldsymbol{r}_1\boldsymbol{r}_3 \\ \boldsymbol{r}_2\boldsymbol{r}_1 & \boldsymbol{r}_2\boldsymbol{r}_2 & \boldsymbol{r}_2\boldsymbol{r}_3 \\ \boldsymbol{r}_3\boldsymbol{r}_1 & \boldsymbol{r}_3\boldsymbol{r}_2 & \boldsymbol{r}_3\boldsymbol{r}_3 \end{vmatrix} = 0. \qquad (*)$$

4. Show that three vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$ are linearly dependent if and only if the condition $(*)$ is fulfilled for them.

5. Show that for any four vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$, $\boldsymbol{r}_4$

$$\begin{vmatrix} \boldsymbol{r}_1\boldsymbol{r}_1 & \boldsymbol{r}_1\boldsymbol{r}_2 & \boldsymbol{r}_1\boldsymbol{r}_3 & \boldsymbol{r}_1\boldsymbol{r}_4 \\ \boldsymbol{r}_2\boldsymbol{r}_1 & \boldsymbol{r}_2\boldsymbol{r}_2 & \boldsymbol{r}_2\boldsymbol{r}_3 & \boldsymbol{r}_2\boldsymbol{r}_4 \\ \boldsymbol{r}_3\boldsymbol{r}_1 & \boldsymbol{r}_3\boldsymbol{r}_2 & \boldsymbol{r}_3\boldsymbol{r}_3 & \boldsymbol{r}_3\boldsymbol{r}_4 \\ \boldsymbol{r}_4\boldsymbol{r}_1 & \boldsymbol{r}_4\boldsymbol{r}_2 & \boldsymbol{r}_4\boldsymbol{r}_3 & \boldsymbol{r}_4\boldsymbol{r}_4 \end{vmatrix} = 0.$$

6. Let l_1, l_2, l_3, and l_4 be four rays emanating from one point, and α_{ij} the angle between the rays l_i and l_j. Show that in this case we have the identity

$$\begin{vmatrix} 1 & \cos\alpha_{12} & \cos\alpha_{13} & \cos\alpha_{14} \\ \cos\alpha_{21} & 1 & \cos\alpha_{23} & \cos\alpha_{24} \\ \cos\alpha_{31} & \cos\alpha_{32} & 1 & \cos\alpha_{34} \\ \cos\alpha_{41} & \cos\alpha_{42} & \cos\alpha_{43} & 1 \end{vmatrix} = 0.$$

Sec. 4-4. The Vector Product of Vectors

The *vector product* of a vector $\boldsymbol{a}$ by a vector $\boldsymbol{b}$ is a third vector $\boldsymbol{a} \times \boldsymbol{b}$ defined in the following way. If at least one of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ is equal to zero or the vectors are parallel, then $\boldsymbol{a} \times \boldsymbol{b} = 0$. In other cases this vector (by its absolute value) is equal to the area of the parallelogram constructed on the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as sides and is

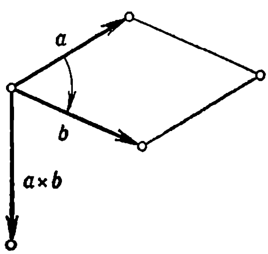


Fig. 58.

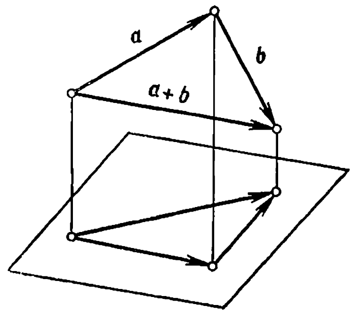


Fig. 59.

directed perpendicular to the plane containing this parallelogram so that the rotation in the direction from $\boldsymbol{a}$ to $\boldsymbol{b}$ and the direction of $\boldsymbol{a} \times \boldsymbol{b}$ form a "right-hand screw" (Fig. 58).

From the definition of the vector product it directly follows:

(1) $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$;

(2) $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta$, *where* θ *is the angle formed by the vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$;

(3) $(\lambda \boldsymbol{a}) \times \boldsymbol{b} = \lambda (\boldsymbol{a} \times \boldsymbol{b})$.

The *projection* of a vector $\boldsymbol{a}$ on a plane is defined as the vector $\boldsymbol{a}'$ whose initial point is the projection of the initial point of the vector $\boldsymbol{a}$ and whose terminal point is the projection of the terminal point of the vector $\boldsymbol{a}$. Obviously, equal vectors have equal projections and the projection of the sum of vectors is equal to the sum of the projections (Fig. 59).

Suppose we have two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Let $\boldsymbol{a}'$ denote the projection of the vector $\boldsymbol{a}$ on the plane perpendicular to the vector $\boldsymbol{b}$ (Fig. 60). Then

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a}' \times \boldsymbol{b}.$$

The proof is obvious. It is sufficient to mention that the vectors $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{a}' \times \boldsymbol{b}$ have equal absolute values and are in the same direction.

The vector product possesses a distributive property, i.e. for any three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$

$$(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}. \qquad (*)$$

The assertion is obvious if $\boldsymbol{c} = 0$. It is then obvious that the equality $(*)$ is sufficient to be proved for the case

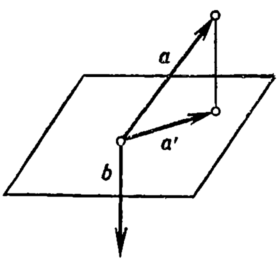


Fig. 60.

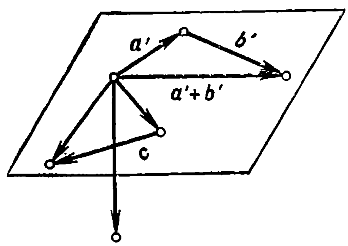


Fig. 61.

$|\boldsymbol{c}| = 1$, since in the general case it will then follow the above mentioned property (3).

So, let $|\boldsymbol{c}| = 1$, and let $\boldsymbol{a}'$ and $\boldsymbol{b}'$ denote the projections of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ on the plane perpendicular to the vector $\boldsymbol{c}$ (Fig. 61). Then the vectors $\boldsymbol{a}' \times \boldsymbol{c}$, $\boldsymbol{b}' \times \boldsymbol{c}$ and $(\boldsymbol{a}' + \boldsymbol{b}') \times \boldsymbol{c}$ are obtained from the vectors $\boldsymbol{a}'$, $\boldsymbol{b}'$, and $\boldsymbol{a}' + \boldsymbol{b}'$, respectively, by a rotation through an angle of 90°. Consequently,

$$(\boldsymbol{a}' + \boldsymbol{b}') \times \boldsymbol{c} = \boldsymbol{a}' \times \boldsymbol{c} + \boldsymbol{b}' \times \boldsymbol{c}.$$

And since

$$\boldsymbol{a}' \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c}, \quad \boldsymbol{b}' \times \boldsymbol{c} = \boldsymbol{b} \times \boldsymbol{c},$$
$$(\boldsymbol{a}' + \boldsymbol{b}') \times \boldsymbol{c} = (\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c},$$

we get

$$(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c},$$

which was required to be proved.

Let us mention the following simple identity which is true for any vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$(\boldsymbol{a} \times \boldsymbol{b})^2 = \boldsymbol{a}^2\boldsymbol{b}^2 - (\boldsymbol{ab})^2.$$

Indeed, if θ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then this identity expresses that

$$(|\boldsymbol{a}|\,|\boldsymbol{b}|\sin\theta)^2 = |\boldsymbol{a}|^2\,|\boldsymbol{b}|^2 - (|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta)^2$$

and, consequently, is obvious.

EXERCISES

1. If the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to the vector $\boldsymbol{c}$, then

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = 0.$$

Show this.

2. If the vector $\boldsymbol{b}$ is perpendicular to $\boldsymbol{c}$, and the vector $\boldsymbol{a}$ is parallel to the vector $\boldsymbol{c}$, then

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{b}\,(\boldsymbol{ac}).$$

Show this.

3. For an arbitrary vector $\boldsymbol{a}$ and a vector $\boldsymbol{b}$ perpendicular to $\boldsymbol{c}$

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{b}\,(\boldsymbol{ac}).$$

Show this.

4. Show that for any three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{b}\,(\boldsymbol{ac}) - \boldsymbol{a}\,(\boldsymbol{bc}).$$

5. Find the area of the base of a triangular pyramid whose lateral edges are equal to l, the vertex angles being equal to α, β, γ.

Sec. 4-5. The Triple Product of Vectors

The *triple (scalar) product* of vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ is the number

$$(\boldsymbol{abc}) = (\boldsymbol{a} \times \boldsymbol{b})\,\boldsymbol{c}. \qquad (*)$$

Obviously, *the triple product is equal to zero if and only if one of the vectors is equal to zero or all three vectors are parallel to one plane.*

The numerical value of the triple product of non-zero vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ which are not parallel to one plane is equal to the volume of the parallelepiped of which the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are coterminal sides (Fig. 62).

Indeed, $\boldsymbol{a} \times \boldsymbol{b} = S\,\boldsymbol{e}$, where S is the area of the base of the parallelepiped constructed on the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{e}$ is the unit vector perpendicular to the base. Further, $\boldsymbol{ec}$ is equal up to a sign to the altitude of the parallelepided dropped onto the mentioned base. Consequently, up to a sign, $(\boldsymbol{abc})$ is equal to the volume of the parallelepiped constructed on the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

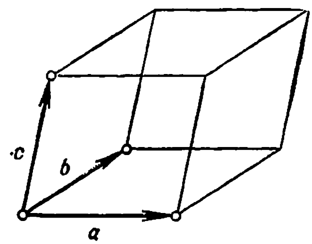


Fig. 62.

The triple product possesses the following property

$$(\boldsymbol{abc}) = \boldsymbol{a}\,(\boldsymbol{b} \times \boldsymbol{c}). \quad (**)$$

It is sufficient to note that the right-hand and the left-hand members are equal by absolute value and have the same sign. From the definition (*) of the triple product and the property (**) it follows that an *interchange of any two factors reverses the sign of the triple product. In particular, the triple product is equal to zero if two factors are equal to each other.*

EXERCISES

1. Noting that

$$((\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c})\,\boldsymbol{d} = (\boldsymbol{a} \times \boldsymbol{b})\,(\boldsymbol{c} \times \boldsymbol{d}),$$

derive the identity

$$(\boldsymbol{a} \times \boldsymbol{b})\,(\boldsymbol{c} \times \boldsymbol{d}) = \begin{vmatrix} \boldsymbol{ac} & \boldsymbol{ad} \\ \boldsymbol{bc} & \boldsymbol{bd} \end{vmatrix}.$$

2. With the aid of the identity

$$(\boldsymbol{a} \times \boldsymbol{b})\,(\boldsymbol{c} \times \boldsymbol{b}) = (\boldsymbol{ac})\,\boldsymbol{b}^2 - (\boldsymbol{ab})\,(\boldsymbol{bc})$$

derive the formula of spherical trigonometry

$$\sin\alpha \sin\gamma \cos B = \cos\beta - \cos\gamma\cos\alpha,$$

where α, β, γ are the sides of a triangle on the unit sphere, and B is the angle of this triangle opposite to the side β.

3. Derive the identity

$$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = \boldsymbol{b}\,(\boldsymbol{acd}) - \boldsymbol{a}\,(\boldsymbol{bcd}).$$

4. Show that for any four vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$

$$\boldsymbol{b}\,(\boldsymbol{acd}) - \boldsymbol{a}\,(\boldsymbol{bcd}) + \boldsymbol{d}\,(\boldsymbol{cab}) - \boldsymbol{c}\,(\boldsymbol{dab}) = 0.$$

5. Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ be any three vectors satisfying the condition

$$(\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3) \neq 0.$$

Then any vector $\boldsymbol{r}$ allows the representation

$$\boldsymbol{r} = \frac{(\boldsymbol{re}_2\boldsymbol{e}_3)\,\boldsymbol{e}_1}{(\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3)} + \frac{(\boldsymbol{re}_3\boldsymbol{e}_1)\,\boldsymbol{e}_2}{(\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3)} + \frac{(\boldsymbol{re}_1\boldsymbol{e}_2)\,\boldsymbol{e}_3}{(\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3)}.$$

Show this.

6. Show that the solution of the following system of vector equations

$$(\boldsymbol{rab}) = \gamma, \quad (\boldsymbol{rbc}) = \alpha, \quad (\boldsymbol{rca}) = \beta,$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are the given vectors satisfying the condition

$$(\boldsymbol{abc}) \neq 0,$$

and $\boldsymbol{r}$ is the required vector, can be written in the form

$$\boldsymbol{r} = \frac{1}{(\boldsymbol{abc})}\,(\boldsymbol{a}\alpha + \boldsymbol{b}\beta + \boldsymbol{c}\gamma).$$

7. Show that if $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ and $\boldsymbol{r}$ are any four vectors satisfying the only condition $(\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3) \neq 0$, then the following identity takes place

$$\boldsymbol{r} = \frac{(\boldsymbol{e}_1 \times \boldsymbol{e}_2)(\boldsymbol{re}_3)}{(\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3)} + \frac{(\boldsymbol{e}_2 \times \boldsymbol{e}_3)(\boldsymbol{re}_1)}{(\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3)} + \frac{(\boldsymbol{e}_3 \times \boldsymbol{e}_1)(\boldsymbol{re}_2)}{(\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3)}.$$

8. Show that the solution of the system of vector equations

$$\boldsymbol{ax} = \alpha, \quad \boldsymbol{bx} = \beta, \quad \boldsymbol{cx} = \gamma,$$

where a, b, c are the given vectors and x is the required vector satisfying the condition $(abc) \neq 0$, can be written in the form

$$x = \frac{(a \times b)\gamma + (b \times c)\alpha + (c \times a)\beta}{(abc)}.$$

Sec. 4-6. The Coordinates of a Vector Relative to a Given Basis

Let e_1, e_2, e_3 be any non-zero vectors not parallel to one plane. Then any vector allows a unique representation of the form

$$r = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3. \qquad (*)$$

The numbers λ_1, λ_2, λ_3 are called the *coordinates of the vector* r relative to the *basis* e_1, e_2, e_3.

Let us first prove that the representation $(*)$ is unique. Suppose there exists another representation:

$$r = \lambda_1' e_1 + \lambda_2' e_2 + \lambda_3' e_3.$$

Then

$$(\lambda_1 - \lambda_1')\, e_1 + (\lambda_2 - \lambda_2')\, e_2 + (\lambda_3 - \lambda_3')\, e_3 = 0.$$

Multiplying this equality scalarly by the vector $e_2 \times e_3$, we get

$$(\lambda_1 - \lambda_1')\,(e_1 e_2 e_3) = 0.$$

Since $(e_1 e_2 e_3) \neq 0$, then $\lambda_1 - \lambda_1' = 0$. Analogously, we conclude that $\lambda_2 - \lambda_2' = 0$, $\lambda_3 - \lambda_3' = 0$. The uniqueness of the representation $(*)$ is proved.

Let us now prove the possibility of the representation $(*)$. Suppose the vector r is parallel to any of the vectors e_1, e_2, e_3, say e_1. Then

$$r = \pm \frac{|r|}{|e_1|} e_1 = \lambda e_1,$$

where the plus sign is taken if the vectors r and e_1 are in the same direction, the minus sign being taken if they are in opposite directions.

Let now the vector r, together with the vectors e_1 and e_2, be parallel to one plane, but is not parallel either to the vector e_1, or to the vector e_2. We draw through the end-points of the vector r straight lines parallel to the vectors e_1 and e_2 (Fig. 63). Then

$$r = r_1 + r_2.$$

But we have proved that

$$r_1 = \lambda_1 e_1, \ r_2 = \lambda_2 e_2.$$

Hence,

$$r = \lambda_1 e_1 + \lambda_2 e_2.$$

Let, finally, the vector r, together with no pair of vectors e_1, e_2; e_2, e_3; e_3, e_1 be not parallel to one plane.

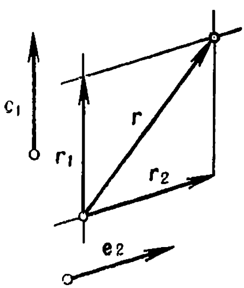


Fig. 63.

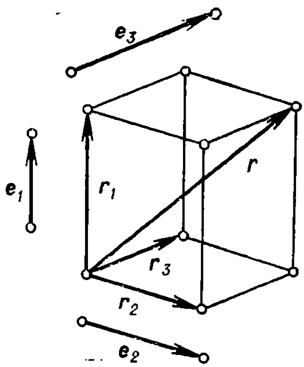


Fig. 64.

We draw through the end-points of the vector r planes parallel to the mentioned pairs of vectors (Fig. 64). Then

$$r = r_1 + r_2 + r_3,$$

and since we proved that

$$r_1 = \lambda_1 e_1, \quad r_2 = \lambda_2 e_2, \quad r_3 = \lambda_3 e_3,$$

we have

$$r = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3.$$

Thus, the possibility of representation of the vector r in the form (*) is proved in all the cases.

The coordinates of a vector have a simple meaning if the basis consists of three pairwise orthogonal unit vectors.

Indeed, multiplying the equality $r = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$ in turn by e_1, e_2, e_3 and noting that $e_1^2 = e_2^2 = e_3^2 = 1$, and $e_1 e_2 = e_2 e_3 = e_3 e_1 = 0$, we get

$$\lambda_1 = re_1, \quad \lambda_2 = re_2, \quad \lambda_3 = re_3.$$

Let r be a vector with the coordinates λ_1, λ_2, λ_3, and r' a vector with the coordinates λ_1', λ_2', λ_3'. We then find the coordinates of the vector $r \pm r'$. We have

$$r = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3,$$
$$r' = \lambda_1' e_1 + \lambda_2' e_2 + \lambda_3' e_3.$$

Whence $r \pm r' = (\lambda_1 \pm \lambda_1') e_1 + (\lambda_2 \pm \lambda_2') e_2 + (\lambda_3 \pm \lambda_3') e_3$. Hence, $\lambda_1 \pm \lambda_1'$, $\lambda_2 \pm \lambda_2'$, $\lambda_3 \pm \lambda_3'$ are the *coordinates of the vector* $r \pm r'$.

We show in a similar way that the vector λr *has the coordinates* $\lambda\lambda_1$, $\lambda\lambda_2$, $\lambda\lambda_3$. Hence it follows that *parallel vectors have proportional coordinates.*

Let the basis e_1, e_2, e_3 consist of three pairwise perpendicular unit vectors whose triple product is equal to $+1$. *We now find the scalar product of the vectors* r and r' *with the coordinates* λ_1, λ_2, λ_3 and λ_1', λ_2', λ_3', *respectively.*

We have

$$r = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3, \quad r' = \lambda_1' e_1 + \lambda_2' e_2 + \lambda_3' e_3. \quad (**)$$

Whence, taking into account that $e_1^2 = e_2^2 = e_3^2 = 1$, $e_1 e_2 = e_2 e_3 = e_3 e_1 = 0$, we get

$$rr' = \lambda_1 \lambda_1' + \lambda_2 \lambda_2' + \lambda_3 \lambda_3'.$$

Let us *find the coordinates* of the vector $r \times r'$. Taking into consideration the representations (**) for the vectors r, r' and the relations $e_1 \times e_2 = e_3$, $e_2 \times e_3 = e_1$, $e_3 \times e_1 = e_2$, we obtain

$$r \times r' = (\lambda_2 \lambda_3' - \lambda_3 \lambda_2') e_1 + (\lambda_3 \lambda_1' - \lambda_1 \lambda_3') e_2 + (\lambda_1 \lambda_2' - \lambda_2 \lambda_1') e_3.$$

Hence the coordinates of the vector $r \times r'$:

$$\begin{vmatrix} \lambda_2 & \lambda_3 \\ \lambda_2' & \lambda_3' \end{vmatrix}, \quad \begin{vmatrix} \lambda_3 & \lambda_1 \\ \lambda_3' & \lambda_1' \end{vmatrix}, \quad \begin{vmatrix} \lambda_1 & \lambda_2 \\ \lambda_1' & \lambda_2' \end{vmatrix}.$$

We finally compute the triple product of the vectors

$$r\ (\lambda_1,\ \lambda_2,\ \lambda_3), \quad r'\ (\lambda_1',\ \lambda_2',\ \lambda_3'), \quad r''\ (\lambda_1'',\ \lambda_2'',\ \lambda_3'').$$

We have

$$(rr'r'') = (r \times r')\, r'' =$$

$$= \begin{vmatrix} \lambda_2 & \lambda_3 \\ \lambda_2' & \lambda_3' \end{vmatrix} \lambda_1'' + \begin{vmatrix} \lambda_3 & \lambda_1 \\ \lambda_3' & \lambda_1' \end{vmatrix} \lambda_2'' + \begin{vmatrix} \lambda_1 & \lambda_2 \\ \lambda_1' & \lambda_2' \end{vmatrix} \lambda_3'' =$$

$$= \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1' & \lambda_2' & \lambda_3' \\ \lambda_1'' & \lambda_2'' & \lambda_3'' \end{vmatrix}.$$

EXERCISES

1. Show that the coordinates of the vector r relative to the basis e_1, e_2, e_3 are given by the equalities

$$\lambda_1 = \frac{(re_2e_3)}{(e_1e_2e_3)}, \quad \lambda_2 = \frac{(re_3e_1)}{(e_1e_2e_3)}, \quad \lambda_3 = \frac{(re_1e_2)}{(e_1e_2e_3)}.$$

2. Show that the coordinates of the vector r relative to the basis $(e_2 \times e_3)$, $(e_3 \times e_1)$, $(e_1 \times e_2)$ are respectively equal to

$$\lambda_1 = \frac{re_1}{(e_1e_2e_3)}, \quad \lambda_2 = \frac{re_2}{(e_1e_2e_3)}, \quad \lambda_3 = \frac{re_3}{(e_1e_2e_3)}.$$

3. Decomposing the vectors a, b, c, on the orthogonal basis, prove the identity

$$(abc)^2 = \begin{vmatrix} aa & ab & ac \\ ba & bb & bc \\ ca & cb & cc \end{vmatrix},$$

using the determinant multiplication theorem.

4. Prove the identity

$$(a \times b,\ b \times c,\ c \times a) = (abc)^2.$$

5. Show that the volume of a triangular pyramid with the lateral edges a, b, c and face angles α, β, γ is

$$V=\frac{1}{6}abc\begin{vmatrix}1 & \cos\gamma & \cos\beta\\ \cos\gamma & 1 & \cos\alpha\\ \cos\beta & \cos\alpha & 1\end{vmatrix}^{1/2}.$$

6. Derive the formula for the volume of a triangular pyramid with the lateral edges a, b, c and the dihedral angles at these edges A, B, C.

Chapter 5

Rectangular Cartesian Coordinates in Space

Sec. 5-1. Cartesian Coordinates

Let us draw from an arbitrary point O in space three straight lines Ox, Oy, Oz not lying in one plane, and lay off on each of them from the point O three non-zero vectors e_x, e_y, e_z (Fig. 65). According to Sec. 4-6, any vector $\overrightarrow{OA}$ allows a unique representation of the form

$$\overrightarrow{OA} = xe_x + ye_y + ze_z.$$

The *numbers* x, y, z are called the *Cartesian coordinates* of a point A.

The straight lines Ox, Oy, Oz are termed the *coordinate axes*: Ox is the x-axis, Oy is the y-axis, and Oz is the z-axis. The planes Oxy, Oyz, Oxz are called the *coordinate planes*: Oxy is the xy-plane, Oyz is the yz-plane, and Oxz is the xz-plane.

Each of the coordinate axes is divided by the point O (i.e. by the *origin of coordinates*) into two semi-axes. Those of the semi-axes whose directions coincide with the directions of the vectors e_x, e_y, e_z are said to be *positive*, the others being *negative*. The coordinate system thus obtained is called *right-handed* if $(e_x e_y e_z) > 0$, and *left-handed* if $(e_x e_y e_z) < 0$.

Geometrically the coordinates of the point A are obtained in the following way. We draw through the point A a plane parallel to the yz-plane. It intersects the x-axis at a point A_x (Fig. 66). Then the absolute value of the coordinate x of the point A is equal to the length of the line segment OA_x as measured by the unit length $|e_x|$.

It is positive if A_x belongs to the positive semi-axis x, and is negative if A_x belongs to the negative semi-axis x. To make sure of this it is sufficient to recall how the coordinates of the vector $\overrightarrow{OA}$ relative to the basis e_x, e_y, e_z are determined. The other two coordinates of the point (y and z) are determined by a similar construction.

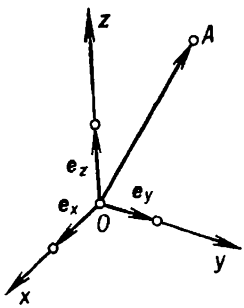


Fig. 65.

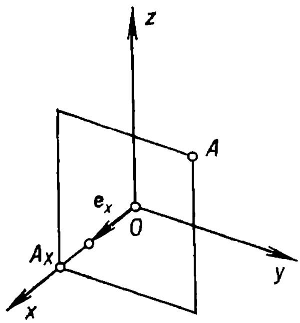


Fig. 66.

If the coordinate axes are mutually perpendicular, and e_x, e_y, e_z are the unit vectors, then the coordinates are called the *rectangular Cartesian coordinates.*

Cartesian coordinates on the plane are introduced in a similar way. Namely, we draw from the point O (i.e. from the origin of coordinates) two arbitrary straight lines Ox and Oy (the coordinate axes) and lay off on each axis (from the point O) a non-zero vector. Thus we obtain the vectors e_x and e_y. The Cartesian coordinates of an arbitrary point A are then determined as the coordinates of the vector $\overrightarrow{OA}$ relative to the basis e_x, e_y.

Obviously, if the coordinate axes are mutually perpendicular, and e_x, e_y are unit vectors, then the coordinates defined in this way coincide with those introduced in Sec. 1-1 and are called the *rectangular Cartesian coordinates.*

Below, as a rule, we shall use the rectangular Cartesian coordinates. If otherwise, each case will be supplied with a special mention.

EXERCISES

1. Where are the points in space located if: (a) $x = 0$; (b) $y = 0$; (c) $z = 0$; (d) $x = 0$, $y = 0$; (e) $y = 0$, $z = 0$; (f) $z = 0$, $x = 0$?

2. How many points in space satisfy the following conditions

$$|x| = a, \quad |y| = b, \quad |z| = c, \quad \text{if} \quad abc \neq 0?$$

3. Where are the points in space situated if

$$|x| < a, \quad |y| < b, \quad |z| < c?$$

4. Let A be a vertex of a parallelepiped, A_1, A_2, A_3 the vertices adjacent to A, i.e. the end-points of the edges emanating from A. Find the coordinates of all the vertices of the parallelepiped, taking the vertex A for the origin and the vertices A_1, A_2, A_3 for the end-points of the basis vectors.

5. Find the coordinates of the point into which the point (x, y, z) goes when rotated about the straight line joining the point A_0 (a, b, c) to the origin through an angle of $\alpha = \pi/2$. The coordinate system is rectangular.

6. Solve Exercise 5 for an arbitrary α.

Sec. 5-2. Elementary Problems of Solid Analytic Geometry

Let there be introduced in space Cartesian coordinates xyz and let A_1 (x_1, y_1, z_1) and A_2 (x_2, y_2, z_2) be two arbitrary points in space. *Find the coordinates of the point A which divides the line segment A_1A_2 in the ratio $\lambda_1 : \lambda_2$* (Fig. 67).

The vectors $\overrightarrow{A_1A}$ and $\overrightarrow{AA_2}$ are in the same direction, and their absolute values are as $\lambda_1 : \lambda_2$. Consequently,

$$\lambda_2\overrightarrow{A_1A} - \lambda_1\overrightarrow{AA_2} = 0,$$

or

$$\lambda_2 (\overrightarrow{OA} - \overrightarrow{OA_1}) - \lambda_1 (\overrightarrow{OA_2} - \overrightarrow{OA}) = 0.$$

Whence

$$\overrightarrow{OA} = \frac{\lambda_2 \overrightarrow{OA_1} + \lambda_1 \overrightarrow{OA_2}}{\lambda_1 + \lambda_2}.$$

Since the coordinates of the point A (x, y, z) are the same as the coordinates of the vector $\overrightarrow{OA}$, we have

$$x = \frac{\lambda_2 x_1 + \lambda_1 x_2}{\lambda_1 + \lambda_2},$$

$$y = \frac{\lambda_2 y_1 + \lambda_1 y_2}{\lambda_1 + \lambda_2},$$

$$z = \frac{\lambda_2 z_1 + \lambda_1 z_2}{\lambda_1 + \lambda_2}.$$

Let the coordinate system be rectangular. *Express the distance between the points A_1 and A_2 in terms of their coordinates.*

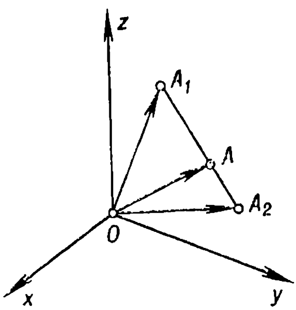


Fig. 67.

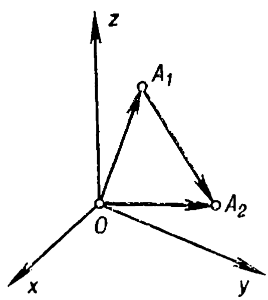


Fig. 68.

The distance between the points A_1 and A_2 is equal to the absolute value of the vector $\overrightarrow{A_1A_2}$ (Fig. 68). We have

$$\overrightarrow{A_1A_2} = \overrightarrow{OA_2} - \overrightarrow{OA_1} = e_x (x_2 - x_1) + e_y (y_2 - y_1) + e_z (z_2 - z_1).$$

Whence

$$(A_1A_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

Express the area of a triangle in the xy-plane in terms of the coordinates of its vertices: $A_1(x_1, y_1, 0)$, $A_2(x_2, y_2, 0)$, and $A_3(x_3, y_3, 0)$.

The absolute value of the vector $\overrightarrow{A_1A_2} \times \overrightarrow{A_1A_3}$ is equal to twice the area of the triangle $A_1A_2A_3$;

$$\overrightarrow{A_1A_2} \times \overrightarrow{A_1A_3} = e_z \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}.$$

Consequently, the area of the triangle

$$S = \frac{1}{2} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}.$$

Express the volume of a tetrahedron $A_1A_2A_3A_4$ *in terms of the coordinates of its vertices.*

The triple scalar product of the vectors $\overrightarrow{A_1A_2}$, $\overrightarrow{A_1A_3}$, $\overrightarrow{A_1A_4}$ is equal (up to a sign) to the volume of the parallelepiped constructed on these vectors and, consequently, to six times the volume of the tetrahedron $A_1A_2A_3A_4$. Hence

$$V = \frac{1}{6} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix}.$$

EXERCISES

1. Find the distance between two points expressed in terms of Cartesian coordinates if the positive semi-axes form pairwise the angles α, β, γ, and e_x, e_y, e_z are unit vectors.

2. Find the centre of a sphere circumscribed about a tetrahedron with the vertices $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, $(0, 0, 0)$.

3. Prove that the straight lines joining the mid-points of the opposite edges of a tetrahedron intersect at one point. Express the coordinates of this point in terms of the coordinates of the vertices of the tetrahedron.

4. Prove that the straight lines joining the vertices of a tetrahedron to the centroids of the opposite faces intersect at one point. Express its coordinates in terms of the coordinates of the vertices of the tetrahedron.

5. Let $A_i\ (x_i,\ y_i,\ z_i)$ be the vertices of a tetrahedron. Show that the points with the coordinates

$$\begin{aligned} x &= \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4, \\ y &= \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \lambda_4 y_4, \\ z &= \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 + \lambda_4 z_4 \end{aligned}$$

are situated inside the tetrahedron if $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, $\lambda_4 > 0$, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$.

6. Express the area of an oblique triangle in terms of the coordinates of its vertices. The coordinate system is rectangular.

7. Show that the formula for computing the volume of a tetrahedron expressed in terms of the coordinates of its vertices is reduced to the form

$$V = \frac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

8. For four points $A_i\ (x_i,\ y_i,\ z_i)$ to lie in one plane it is necessary and sufficient that

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

Prove this.

Sec. 5-3. Equations of a Surface and a Curve in Space

Suppose we have a surface (Fig. 69).

The equation

$$f(x,\ y,\ z) = 0 \qquad (*)$$

is called the *equation of a surface in implicit form* if the coordinates of any point of the surface satisfy this equation. And conversely, any three numbers x, y, z satisfying the equation (*) represent the coordinates of one of the points of the surface.

The system of equations

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v), \qquad (**)$$

specifying the coordinates of the points of the surface as a function of two parameters (u, v) is called the *parametric equation of a surface.*

Eliminating the parameters u, v from the system (**), we can obtain the implicit equation of a surface.

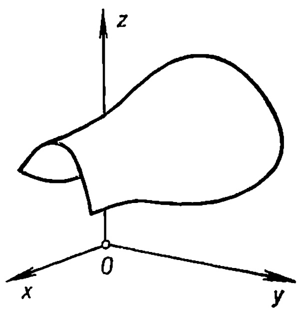


Fig. 69.

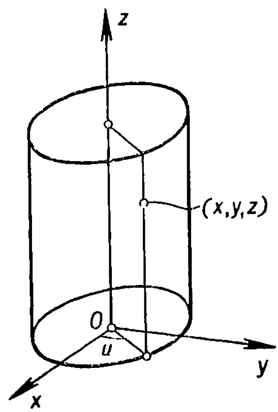


Fig. 70.

Form the equation of an arbitrary sphere in the rectangular Cartesian coordinates xyz.

Let (x_0, y_0, z_0) be the centre of the sphere, and R its radius. Each point (x, y, z) of the sphere is located at a distance R from the centre, and, consequently, satisfies the equation

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - R^2 = 0 \qquad (***)$$

Conversely, any point (x, y, z) satisfying the equation (***) is found at a distance R from (x_0, y_0, z_0) and, consequently, belongs to the sphere. According to the definition, the equation (***) is the equation of a sphere.

Form the equation of a circular cylinder with the axis Oz and radius R (Fig. 70).

Let us take the coordinate z (v) and the angle (u) formed by the plane passing through the z-axis and the point (x, y, z) with the xz-plane as the parameters u, v, characterizing the position of the point (x, y, z) on the cylinder. We then get

$$x = R\cos u, \quad y = R\sin u, \quad z = v,$$

which is the required *equation of the cylinder in parametric form*.

Squaring the first two equations and adding termwise, we get the *equation of the cylinder in implicit form*:

$$x^2 + y^2 = R^2.$$

Suppose we have a curve in space. The system of equations

$$f_1(x, y, z) = 0, \quad f_2(x, y, z) = 0$$

is called the *equation of a curve in implicit form* if the coordinates of each point of the curve satisfy both equations. And conversely, any three numbers satisfying both equations represent the coordinates of some point on the curve.

A system of equations

$$x = \varphi_1(t), \quad y = \varphi_2(t), \quad z = \varphi_3(t),$$

specifying the coordinates of points of the curve as a function of some parameter (t) is termed the *equation of a curve in parametric form*.

Two surfaces intersect, as a rule, along a curve. Obviously, if the surfaces are specified by the equations $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$, then the curve along which they intersect is represented by a system of equations

$$f_1(x, y, z) = 0, \quad f_2(x, y, z) = 0.$$

Form the equation of an arbitrary circle in space. Any circle can be represented as an intersection of two spheres. Consequently, any circle can be specified by a system of equations

$$\left.\begin{aligned} (x-a_1)^2 + (y-b_1)^2 + (z-c_1)^2 - R_1^2 = 0, \\ (x-a_2)^2 + (y-b_2)^2 + (z-c_2)^2 - R_2^2 = 0. \end{aligned}\right\}$$

As a rule, a curve and a surface intersect at separate points. If the surface is specified by the equation $f(x, y, z) = 0$, and the curve by the equations $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$, then the points of intersection of the curve and the surface satisfy the following system of equations:

$$f(x, y, z) = 0, \quad f_1(x, y, z) = 0, \quad f_2(x, y, z) = 0.$$

Solving this system, we find the coordinates of the points of intersection.

EXERCISES

1. Show that the surface represented by the equation

$$x^2 + y^2 + z^2 + 2ax + 2by + 2cz + d = 0,$$

is a sphere if $a^2 + b^2 + c^2 - d > 0$. Find the coordinates of its centre and radius.

2. A circle is specified by the intersection of two spheres

$$\left.\begin{aligned} f_1(x, y, z) &= x^2 + y^2 + z^2 + 2a_1x + 2b_1y + 2c_1z + d_1 = 0, \\ f_2(x, y, z) &= x^2 + y^2 + z^2 + 2a_2x + 2b_2y + 2c_2z + d_2 = 0. \end{aligned}\right\}$$

Show that any sphere passing through this circle can be represented by the equation

$$\lambda_1 f_1(x, y, z) + \lambda_2 f_2(x, y, z) = 0.$$

3. Show that the surface specified by an equation of the form $\varphi(x, y) = 0$ is cylindrical. It is generated by straight lines parallel to the z-axis.

4. Form the equation of a right circular cone with the axis Oz, vertex O, and the vertex angle equal to 2α.

5. Form the equation of a surface described by the midpoint of a line segment whose end-points belong to the curves γ_1 and γ_2

$$\gamma_1: \left.\begin{aligned} z &= ax^2, \\ y &= 0, \end{aligned}\right\} \qquad \gamma_2: \left.\begin{aligned} z &= by^2, \\ x &= 0. \end{aligned}\right\}.$$

6. Form the equation of a surface generated by a straight line which intersects the curves γ_1 and γ_2, re-

maining all the time parallel to the yz-plane:

$$\gamma_1: \left.\begin{array}{l} z=f(x), \\ y=a, \end{array}\right\} \quad \gamma_2: \left.\begin{array}{l} z=\varphi(x), \\ y=b \end{array}\right\} \quad (a \neq b)$$

7. Show that the curve

$$z = \varphi(x), \quad y = 0 \quad (x > 0),$$

when revolving about the z-axis, generates a surface specified by the equation

$$z=\varphi\left(\sqrt{x^2+y^2}\right).$$

8. Show that a cylindrical surface, with the generatrix parallel to the z-axis, passing through the curve

$$z = f(x), \quad z = \varphi(y),$$

is specified by the equation

$$f(x) - \varphi(y) = 0.$$

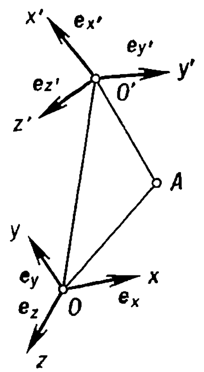


Fig. 71.

Sec. 5-4. Transformation of Coordinates

Let there be introduced in space two Cartesian systems of coordinates xyz and $x'y'z'$ (Fig. 71). *Express the coordinates of an arbitrary point A in the coordinate system $x'y'z'$ in terms of its coordinates in the coordinate system xyz.*

We have

$$\overrightarrow{O'A}=x'e_{x'}+y'e_{y'}+z'e_{z'},$$

$$\overrightarrow{O'O}=x_0'e_{x'}+y_0'e_{y'}+z_0'e_{z'},$$

$$\overrightarrow{OA}=xe_x+ye_y+ze_z,$$

$$\overrightarrow{O'A}=\overrightarrow{O'O}+\overrightarrow{OA}=(x_0'e_{x'}+y_0'e_{y'}+z_0'e_{z'})+(xe_x+ye_y+ze_z).$$

The vectors e_x, e_y, e_z allow a unique representation in terms of the vectors $e_{x'}$, $e_{y'}$, $e_{z'}$:

$$\left.\begin{aligned} e_x &= \alpha_{11}e_{x'} + \alpha_{12}e_{y'} + \alpha_{13}e_{z'}, \\ e_y &= \alpha_{21}e_{x'} + \alpha_{22}e_{y'} + \alpha_{23}e_{z'}, \\ e_z &= \alpha_{31}e_{x'} + \alpha_{32}e_{y'} + \alpha_{33}e_{z'}, \end{aligned}\right\} \qquad (*)$$

where α_{ij} are the coordinates of the vectors e_x, e_y, e_z relative to the basis $e_{x'}$, $e_{y'}$, $e_{z'}$.

Substituting these expressions into the formula for $\overrightarrow{O'A}$, we get

$$\begin{aligned} \overrightarrow{O'A} = (x_0' + \alpha_{11}x + \alpha_{21}y + \alpha_{31}z)\, e_{x'} &+ \\ + (y_0' + \alpha_{12}x + \alpha_{22}y + \alpha_{32}z)\, e_{y'} &+ \\ + (z_0' + \alpha_{13}x + \alpha_{23}y + \alpha_{33}z)\, e_{z'}&, \end{aligned}$$

where the expressions in parentheses are the coordinates of the vector $\overrightarrow{O'A}$ relative to the basis $e_{x'}$, $e_{y'}$, $e_{z'}$, i.e. the coordinates of the point A in the system $x'y'z'$. We get the required formulas:

$$\left.\begin{aligned} x' &= \alpha_{11}x + \alpha_{21}y + \alpha_{31}z + x_0', \\ y' &= \alpha_{12}x + \alpha_{22}y + \alpha_{32}z + y_0', \\ z' &= \alpha_{13}x + \alpha_{23}y + \alpha_{33}z + z_0'. \end{aligned}\right\} \qquad (**)$$

The coefficients of these formulas have the following meaning: α_{11}, α_{12}, α_{13} are the coordinates of the vector e_x relative to the basis $e_{x'}$, $e_{y'}$, $e_{z'}$; α_{21}, α_{22}, α_{23} the coordinates of the vector e_y; α_{31}, α_{32}, α_{33} the coordinates of the vector e_z; x_0', y_0', z_0' the coordinates of the point O in the coordinate system $x'y'z'$.

We note that the determinant

$$\Delta = \begin{vmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{vmatrix} \neq 0$$

Indeed, one can directly check that

$$(e_x e_y e_z) = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} (e_{x'} e_{y'} e_{z'}).$$

Since $(e_x e_y e_z) \neq 0$, then $\Delta \neq 0$.

For all systems of coordinates $x'y'z'$ which can be continuously transformed into one another the determinant Δ has one and the same sign. (The continuity of changing a system of coordinates is understood as the continuity of changing the origin of coordinates O' and the basis $e_{x'}$, $e_{y'}$, $e_{z'}$.) Indeed, since $(e_x e_y e_z)$ is non-zero, Δ is also non-zero. Besides, since Δ changes continuously, it cannot attain values of different signs.

If $\Delta \neq 0$, then the system of formulas (**) may always be interpreted as a passage from a coordinate system $x'y'z'$ to the coordinate system xyz whose origin is situated at point (x_0', y_0', z_0') and the basis vectors are expressed in terms of the basis vectors of the system $x'y'z'$ by the formula (*).

If both systems of coordinates xyz and $x'y'z'$ are rectangular, then the coefficients of the formulas (**) satisfy the *orthogonality conditions*

$$\left.\begin{array}{ll} \alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 = 1, & \alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{23} = 0, \\ \alpha_{21}^2 + \alpha_{22}^2 + \alpha_{23}^2 = 1, & \alpha_{21}\alpha_{31} + \alpha_{22}\alpha_{32} + \alpha_{23}\alpha_{33} = 0, \\ \alpha_{31}^2 + \alpha_{32}^2 + \alpha_{33}^2 = 1, & \alpha_{31}\alpha_{11} + \alpha_{32}\alpha_{12} + \alpha_{33}\alpha_{13} = 0, \end{array}\right\} \qquad (***)$$

which are obtained if use is made of the formulas (*) and the following relationships

$$e_x^2 = e_y^2 = e_z^2 = 1, \qquad e_x e_y = e_y e_z = e_z e_x = 0,$$
$$e_{x'}^2 = e_{y'}^2 = e_{z'}^2 = 1, \quad e_{x'} e_{y'} = e_{y'} e_{z'} = e_{z'} e_{x'} = 0.$$

Conversely, if the conditions (***) are fulfilled, then the formulas (**) can always be interpreted as a passage from a rectangular coordinate system $x'y'z'$ to the system of rectangular coordinates xyz whose origin is located at point (x_0', y_0', z_0') and the basis vectors are specified by the formula (*). By virtue of the conditions (***) the basis vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$ are unit vectors which are perpendicular pairwise.

In the case of rectangular Cartesian coordinates xyz and $x'y'z'$ we have $\Delta = \pm 1$, where $\Delta = +1$ if one system of coordinates can be brought in coincidence with the other system by motion. If it cannot be done by motion without *mirror reflection*, then $\Delta = -1$.

EXERCISES

1. What will be the formulas for transforming the coordinates if the xy-plane coincides with the $x'y'$-plane?

2. It is known that in a certain system of coordinates the equation

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{31}zx = c$$

specifies a sphere. Find the angles between the coordinate axes.

3. Suppose we have two systems of coordinates xyz and $x'y'z'$ with a common origin O. Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ be the basis of the first system, and $\boldsymbol{e}_1 \times \boldsymbol{e}_2$, $\boldsymbol{e}_2 \times \boldsymbol{e}_3$, $\boldsymbol{e}_3 \times \boldsymbol{e}_1$ the basis of the second system. Derive the transformation formulas from one system to the other.

4. The transition from one rectangular Cartesian system of coordinates xyz to the other rectangular Cartesian system of coordinates $x'y'z'$ with the same origin can be accomplished in three stages:

$$\text{I} \qquad \left.\begin{aligned} x_1 &= x\cos\varphi - y\sin\varphi, \\ y_1 &= x\sin\varphi + y\cos\varphi, \\ z_1 &= z; \end{aligned}\right\}$$

$$\text{II} \qquad \left.\begin{aligned} x_2 &= x_1, \\ y_2 &= y_1\cos\theta - z_1\sin\theta, \\ z_2 &= y_1\sin\theta + z_1\cos\theta; \end{aligned}\right\}$$

$$\text{III} \qquad \left.\begin{aligned} x' &= x_2\cos\psi - y_2\sin\psi, \\ y' &= x_2\sin\psi + y_2\cos\psi, \\ z' &= z_2. \end{aligned}\right\}$$

The angles φ, θ, ψ are called *Euler's angles*. Find out their geometrical meaning.

Chapter 6

A Plane and a Straight Line

Sec. 6-1. The Equation of a Plane

Form the equation of an arbitrary plane in the rectangular Cartesian coordinates xyz.

Let A_0 (x_0, y_0, z_0) be a point in a plane and $\boldsymbol{n}$ a nonzero vector perpendicular to the plane. Then whatever the point of the plane A (x, y, z) is, the vectors $\overrightarrow{A_0A}$ and $\boldsymbol{n}$ are mutually perpendicular (Fig. 72). Hence,

$$\overrightarrow{A_0A} \cdot \boldsymbol{n} = 0. \qquad (*)$$

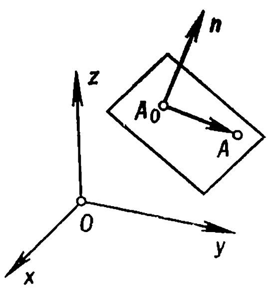


Fig. 72.

Let a, b, c be the coordinates of the vector $\boldsymbol{n}$ with respect to the basis $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$. Then, since $\overrightarrow{A_0A} = \overrightarrow{OA} - \overrightarrow{OA_0}$, it follows from (*)

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \qquad (**)$$

This is the required equation.

Thus, *the equation of any plane is linear relative to the coordinates x, y, z.*

Since the formulas for transition from one Cartesian system of coordinates to another are linear, we may state that *the equation of a plane is linear in any Cartesian system of coordinates* (but not only in a rectangular one).

Let us now show that *any equation of the form*

$$ax + by + cz + d = 0$$

is the equation of a plane,

Let x_0, y_0, z_0 be a solution of the given equation. Then

$$ax_0 + by_0 + cz_0 + d = 0$$

and the equation may be rewritten in the form

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \qquad (***)$$

Let $\boldsymbol{n}$ be a vector with the coordinates a, b, c with respect to the basis $\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$, A_0 a point with the coordinates x_0, y_0, z_0, and A a point with the coordinates x, y, z. Then the equation (***) can be written in the equivalent form

$$\overrightarrow{A_0A} \cdot \boldsymbol{n} = 0.$$

Whence it follows that all points of the plane passing through the point A_0 and perpendicular to the vector $\boldsymbol{n}$ (and only they) satisfy the given equation and, consequently, it is the equation of this plane.

Let us note that the coefficients of x, y, z in the equation of the plane are the coordinates of the vector perpendicular to the plane relative to the basis $\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$.

EXERCISES

1. Form the equation of a plane given two points (x_1, y_1, z_1) and (x_2, y_2, z_2) situated symmetrically about it.

2. Show that the planes

$$ax + by + cz + d_1 = 0,$$
$$ax + by + cz + d_2 = 0, \quad d_1 \neq d_2,$$

are parallel (do not intersect).

3. What is the locus of points whose coordinates satisfy the equation

$$(ax + by + cz + d)^2 - (\alpha x + \beta y + \gamma z + \delta)^2 = 0?$$

4. Show that the curve represented by the equations

$$f(x, y, z) + a_1x + b_1y + c_1z + d_1 = 0,$$
$$f(x, y, z) + a_2x + b_2y + c_2z + d_2 = 0,$$

is a plane one, i.e. all points of this curve belong to a plane.

5. Show that the three planes specified by the equations

$$ax + by + cz + d = 0,$$
$$\alpha x + \beta y + \gamma z + \delta = 0,$$
$$\lambda (ax + by + cz) + \mu (\alpha x + \beta y + \gamma z) + k = 0,$$

have no points in common if $k \neq \lambda d + \mu\delta$.

6. Write the equation of the plane passing through the circle of intersection of the two spheres

$$x^2 + y^2 + z^2 + ax + by + cz + d = 0,$$
$$x^2 + y^2 + z^2 + \alpha x + \beta y + \gamma z + \delta = 0.$$

7. Show that inversion transforms a sphere either into a sphere or into a plane.

8. Show that the equation of any plane passing through the line of intersection of the planes

$$ax + by + cz + d = 0,$$
$$\alpha x + \beta y + \gamma z + \delta = 0,$$

can be represented in the form

$$\lambda (ax + by + cz + d) + \mu (\alpha x + \beta y + \gamma z + \delta) = 0.$$

9. Show that the plane passing through the three given points (x_i, y_i, z_i) $(i = 1, 2, 3)$ is specified by the equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

Sec. 6-2. Special Cases of the Position of a Plane Relative to a Coordinate System

Let us find out the peculiarities of the position of a plane relative to a coordinate system which take place when its equation is of this or that particular form.

1. $a = 0$, $b = 0$. Vector $\boldsymbol{n}$ (perpendicular to the plane) is parallel to the z-axis. The plane is parallel to the

xy-plane. In particular, it coincides with the xy-plane if d is also zero.

2. $b = 0$, $c = 0$. The plane is parallel to the yz-plane and coincides with it if $d = 0$.

3. $c = 0$, $a = 0$. The plane is parallel to the xz-plane and coincides with it if $d = 0$.

4. $a = 0$, $b \neq 0$, $c \neq 0$. Vector $\boldsymbol{n}$ is perpendicular to the x-axis: $\boldsymbol{e}_x\boldsymbol{n} = 0$. The plane is parallel to the x-axis, in particular, it passes through it if $d = 0$.

5. $a \neq 0$, $b = 0$, $c \neq 0$. The plane is parallel to the y-axis and passes through it if $d = 0$.

6. $a \neq 0$, $b \neq 0$, $c = 0$. The plane is parallel to the z-axis and passes through it if $d = 0$.

7. $d = 0$. The plane passes through the origin (whose coordinates 0, 0, 0 satisfy the equation of the plane).

If all the coefficients are non-zero, then the equation may be divided by $-d$. Then, putting

$$-\frac{d}{a} = \alpha, \quad -\frac{d}{b} = \beta, \quad -\frac{d}{c} = \gamma,$$

we get the equation of the plane in the following form:

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1. \qquad (*)$$

The numbers α, β, γ are equal (up to a sign) to the segments intercepted by the plane on the coordinate axes. Indeed, the x-axis ($y = 0$, $z = 0$) is intersected by the plane at point $(\alpha, 0, 0)$, the y-axis at point $(0, \beta, 0)$, and the z-axis at point $(0, 0, \gamma)$. The equation (*) is called the *intercept form of the equation of a plane.*

We conclude with a note that any plane not perpendicular to the xy-plane ($c \neq 0$) may be specified by an equation of the form

$$z = px + qy + l.$$

EXERCISES

1. Find the conditions under which the plane

$$ax + by + cz + d = 0$$

intersects the positive semi-axis x (y, z).

2. Find the volume of the tetrahedron bounded by the coordinate planes and the plane

$$ax + by + cz + d = 0$$

if $abcd \neq 0$.

3. Prove that the points in space for which

$$|x| + |y| + |z| < a,$$

are situated inside an octahedron with centre at the origin and the vertices on the coordinate axes.

4. Given a plane σ by the equation in rectangular Cartesian coordinates

$$ax + by + cz + d = 0.$$

Form the equation of the plane σ' symmetrical to σ about the xy-plane (about the origin O).

5. Given a family of planes depending on a parameter

$$ax + by + cz + d + \lambda (\alpha x + \beta y + \gamma z + \delta) = 0.$$

Find in this family a plane parallel to the z-axis.

6. In the family of planes

$$(a_1x + b_1y + c_1z + d_1) + \lambda (a_2x + b_2y + c_2z + d_2) + \\ + \mu (a_3x + b_3y + c_3z + d_3) = 0$$

find the plane parallel to the xy-plane. The parameters of the family are λ and μ.

Sec. 6-3. The Normal Form of the Equation of a Plane

If a point $A(x, y, z)$ belongs to the plane

$$ax + by + cz + d = 0, \qquad (*)$$

then its coordinates satisfy the equation (*).

Let us find out what geometrical meaning has the expression

$$ax + by + cz + d$$

if the point A does not belong to the plane.

We drop from the point A a perpendicular onto the plane. Let A_0 (x_0, y_0, z_0) be the foot of the perpendicular. Since the point A_0 lies on the plane, then

$$ax_0 + by_0 + cz_0 + d = 0.$$

Whence

$$ax + by + cz + d =$$
$$= a(x - x_0) + b(y - y_0) + c(z - z_0) =$$
$$= \boldsymbol{n} \cdot \overrightarrow{A_0A} = \pm |\boldsymbol{n}| \delta,$$

where $\boldsymbol{n}$ is a vector perpendicular to the plane, with the coordinates a, b, c, and δ is the distance of the point A from the plane.

Thus

$$ax + by + cz + d$$

is positive on one side of the plane, and negative on the other, its absolute value being proportional to the distance of the point A from the plane. The proportionality factor

$$\pm |\boldsymbol{n}| = \pm \sqrt{a^2 + b^2 + c^2}.$$

If in the equation of the plane

$$a^2 + b^2 + c^2 = 1,$$

then

$$ax + by + cz + d$$

will be equal up to a sign to the distance of the point from the plane. In this case the plane is said to be specified by an *equation in the normal form*.

Obviously, to obtain the normal form of the equation of a plane (*), it is sufficient to divide it by

$$\pm \sqrt{a^2 + b^2 + c^2}.$$

EXERCISES

1. The planes specified by the equations in rectangular Cartesian coordinates

$$ax + by + cz + d = 0,$$
$$ax + by + cz + d' = 0,$$

where $d \neq d'$, have no points in common, hence, they are parallel. Find the distance between these planes.

2. The plane

$$ax + by + d = 0$$

is parallel to z-axis. Find the distance of the z-axis from this plane.

3. What is the locus of points whose distance to two given planes are in a given ratio?

4. Form the equations of the planes parallel to the plane

$$ax + by + cz + d = 0$$

and located at a distance δ from it.

5. Show that the points in space satisfying the condition

$$| ax + by + cz + d | < \delta^2,$$

are situated between the parallel planes

$$ax + by + cz + d \pm \delta^2 = 0.$$

6. Given are the equations of the planes containing the faces of a tetrahedron and a point M by its coordinates. How to find out whether or not the point M lies inside the tetrahedron?

7. Derive the formulas for transition to a new system of rectangular Cartesian coordinates $x'y'z'$ if the new coordinate planes are specified in the old system by the equations

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0,$$
$$a_3x + b_3y + c_3z + d_3 = 0.$$

Sec. 6-4. Relative Position of Planes

Suppose we have two planes

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0, \\ a_2x + b_2y + c_2z + d_2 = 0. \end{aligned}\right\} \qquad (*)$$

Find out under which condition these planes are: (a) *parallel,* (b) *mutually perpendicular.*

Since a_1, b_1, c_1 are the coordinates of vector $\boldsymbol{n}_1$ perpendicular to the first plane, and a_2, b_2, c_2 are the coordinates of vector $\boldsymbol{n}_2$ which is perpendicular to the second plane, the planes are parallel if the vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ are parallel, i.e. if their coordinates are proportional:

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}.$$

Moreover, this condition is sufficient for parallelism of the planes if they are not coincident.

For the planes (*) to be mutually perpendicular it is necessary and sufficient that the mentioned vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are mutually perpendicular, which for non-zero vectors is equivalent to the condition

$$\boldsymbol{n}_1\boldsymbol{n}_2 = 0 \quad \text{or} \quad a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

Let the equations (*) specify two arbitrary planes. *Find the angle made by these planes.*

The angle θ between the vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ is equal to one of the angles formed by the planes and is readily found. We have

$$\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = |\boldsymbol{n}_1|\,|\boldsymbol{n}_2| \cos\theta.$$

Whence

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\,\sqrt{a_2^2+b_2^2+c_2^2}}.$$

Suppose we have three different planes:

$$\left.\begin{aligned} a_1x+b_1y+c_1z+d_1=0,\\ a_2x+b_2y+c_2z+d_2=0,\\ a_3x+b_3y+c_3z+d_3=0. \end{aligned}\right\} \qquad (**)$$

The planes (**) either intersect at one point, or are parallel to a straight line, in particular, they pass through a straight line.

If the planes (**) intersect at one point, then the system of equations (**) has a unique solution. As is known from algebra, it will be when and only when the determinant of the system

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$$

It can be explained using another method. If the planes intersect at one point, then the vectors $\boldsymbol{n}_1 (a_1, b_1, c_1)$, $\boldsymbol{n}_2 (a_2, b_2, c_2)$, $\boldsymbol{n}_3 (a_3, b_3, c_3)$ cannot be parallel to one plane (since the planes, intersecting at a point, would then intersect along a straight line), and, consequently, their triple product equal to the determinant Δ is non-zero.

The planes (**) will be parallel to a straight line if $\Delta = 0$ which means that the vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$ are parallel to some plane. If in addition the system (**) is compatible (i.e. has a solution) then the planes intersect along a straight line.

EXERCISES

1. Find the angles formed by the plane

$$ax + by + cz + d = 0$$

and the coordinate axes.

2. Find the angle formed by the plane

$$z = px + qy + l$$

with the xy-plane.

3. Show that the area of a figure F contained in the plane

$$z = px + qy + l$$

and the area of its projection $\bar{F}$ onto the xy-plane are related as follows

$$S(F) = \sqrt{1 + p^2 + q^2}\, S(\bar{F}).$$

4. Under what condition does the plane

$$ax + by + cz + d = 0$$

intersect the x- and y-axes at equal angles? Under what condition does it intersect all three axes?

5. Show that the plane passing through the point (x_0, y_0, z_0) and parallel to the plane

$$ax + by + cz + d = 0,$$

is represented by the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

6. Show that the plane passing through the point (x_0, y_0, z_0) and perpendicular to the planes

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0,$$

is specified by the equation

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

7. Among planes of the pencil

$$\lambda(a_1x + b_1y + c_1z + d_1) + \mu(a_2x + b_2y + c_2z + d_2) = 0$$

find the plane perpendicular to the plane

$$ax + by + cz + d = 0.$$

8. Let

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0,$$
$$a_3x + b_3y + c_3z + d_3 = 0$$

be the equations of three planes not parallel to a straight line. Then any plane passing through the point of intersection of the given planes has the equation of the form:

$$\lambda_1(a_1x + b_1y + c_1z + d_1) + \lambda_2(a_2x + b_2y + c_2z + d_2) + \lambda_3(a_3x + b_3y + c_3z + d_3) = 0.$$

Sec. 6-5. Equations of the Straight Line

Any straight line can be specified as an intersection of two planes. Consequently, *any straight line can be specified by the equations*

$$\left.\begin{aligned} a_1x+b_1y+c_1z+d_1=0, \\ a_2x+b_2y+c_2z+d_2=0, \end{aligned}\right\} \quad (*)$$

the first of which represents one plane and the second the other. Conversely, *any compatible system of two such independent equations represents the equations of a straight line.*

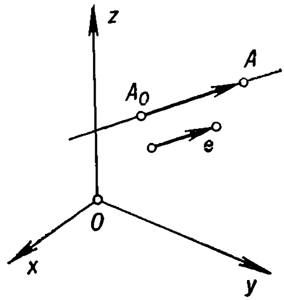


Fig. 73.

Let A_0 (x_0, y_0, z_0) be a fixed point on a straight line, A (x, y, z) an arbitrary point of the straight line, and e (k, l, m) a non-zero vector parallel to the straight line (Fig. 73). Then the vectors $\overrightarrow{A_0A}$ and e are parallel and, consequently, their coordinates are proportional, i.e.

$$\frac{x-x_0}{k}=\frac{y-y_0}{l}=\frac{z-z_0}{m}. \qquad (**)$$

This form of the equation of a straight line is called *canonical*. It represents a particular case of (*), since it allows an equivalent form

$$\frac{x-x_0}{k}=\frac{y-y_0}{l}, \quad \frac{y-y_0}{l}=\frac{z-z_0}{m},$$

corresponding to (*).

Suppose a straight line is represented by the equations (*). Let us form its equation in canonical form. For this purpose it is sufficient to find a point A_0 on the straight line and a vector e parallel to this line.

Any vector e (k, l, m) parallel to the straight line will be parallel to either of the planes (*), and conversely. Consequently, k, l, m satisfy the equations

$$\left.\begin{aligned} a_1k+b_1l+c_1m=0, \\ a_2k+b_2l+c_2m=0. \end{aligned}\right\} \qquad (***)$$

Thus, any solution of the system (*) may be taken as x_0, y_0, z_0 for the canonical equation of the straight line and any solution of (***) as the coefficients k, l, m, for instance

$$k=\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad l=\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \quad m=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

From the equation of a straight line in canonical form we can derive its equations in parametric form. Namely, putting the common value of the three ratios of the canonical equation equal to t, we get

$$x = kt + x_0, \quad y = lt + y_0, \quad z = mt + z_0$$

which are the *parametric equations of a straight line.*

Let us find out what are the peculiarities of the position of a straight line relative to the coordinate system if some of the coefficients of the canonical equation are equal to zero.

Since the vector $\boldsymbol{e}$ (k, l, m) is parallel to the straight line, with $m = 0$ the line is parallel to the xy-plane $(\boldsymbol{e}\boldsymbol{e}_z = 0)$, with $l = 0$ the line is parallel to the xz-plane, and with $k = 0$ it is parallel to the yz-plane.

If $k = 0$ and $l = 0$, then the straight line is parallel to the z-axis ($\boldsymbol{e}$ is parallel to $\boldsymbol{e}_z$); if $l = 0$ and $m = 0$, then it is parallel to the x-axis, and if $k = 0$ and $m = 0$, then the line is parallel to the y-axis.

We conclude with a note that a straight line may be specified by the equations of the form (*) and (**) in Cartesian coordinates in general (and not only in its particular case, i.e. in rectangular Cartesian coordinates).

EXERCISES

1. Under what condition does a straight line represented by the equation in canonical form (**) intersect the x-axis (y-axis, z-axis)? Under what condition is it parallel to the plane xy (yz, zx)?
2. Show that the locus of points equidistant from three pairwise non-parallel planes is a straight line.
3. Show that the locus of points equidistant from the vertices of a triangle is a straight line. Form its equations given the coordinates of the vertices of the triangle.

4. Show that through each point of the surface

$$z = axy$$

there pass two straight lines entirely lying on the surface.

5. If the straight lines specified by the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0, \\ a_2x + b_2y + c_2z + d_2 = 0 \end{aligned}\right\}$$

and

$$\left.\begin{aligned} a_3x + b_3y + c_3z + d_3 = 0, \\ a_4x + b_4y + c_4z + d_4 = 0, \end{aligned}\right\}$$

intersect, then

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

Show this.

Sec. 6-6. Relative Position of a Straight Line and a Plane, of Two Straight Lines.

Suppose we have a straight line and a plane respectively specified by the equations

$$ax + by + cz + d = 0,$$

$$\frac{x-x_0}{k} = \frac{y-y_0}{l} = \frac{z-z_0}{m}.$$

Since the vector (a, b, c) is perpendicular to the plane, and the vector (k, l, m) is parallel to the straight line, then the *straight line and the plane will be parallel if these vectors are mutually perpendicular, i.e. if*

$$ak + bl + cm = 0. \qquad (*)$$

Moreover, if the point (x_0, y_0, z_0) belonging to the straight line satisfies the equation of the plane

$$ax_0 + by_0 + cz_0 + d = 0,$$

then the *straight line lies in the plane.*

The straight line and the plane are mutually perpendicular if the vectors (a, b, c) *and* $(k, l, m,)$ *are parallel, i.e. if*

$$\frac{a}{k}=\frac{b}{l}=\frac{c}{m}. \qquad (**)$$

We can obtain the parallelism and perpendicularity conditions for a straight line and a plane if the straight line is represented by the intersection of the planes

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0.$$

It is sufficient to note that the vector with the coordinates

$$k=\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad l=\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \quad m=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

is parallel to the straight line and make use of the conditions $(*)$ and $(**)$.

Suppose two straight lines are specified by the equations in canonical form

$$\left.\begin{aligned} \frac{x-x'}{k'}=\frac{y-y'}{l'}=\frac{z-z'}{m'}, \\ \frac{x-x''}{k''}=\frac{y-y''}{l''}=\frac{z-z''}{m''}. \end{aligned}\right\} \qquad (***)$$

Since the vector (k', l', m') is parallel to the first line, and the vector (k'', l'', m'') is parallel to the second line, then the *lines are parallel if*

$$\frac{k'}{k''}=\frac{l'}{l''}=\frac{m'}{m''}.$$

In particular, the straight lines coincide if a point of the first line, say (x', y', z'), satisfies the equation of the second line, i.e. if

$$\frac{x'-x''}{k''}=\frac{y'-y''}{l''}=\frac{z'-z''}{m''}.$$

The straight lines are mutually perpendicular if the vectors (k', l', m') *and* (k'', l'', m'') *are mutually perpendicular, i.e. if*

$$k'k'' + l'l'' + m'm'' = 0.$$

If two straight lines are specified by equations of one of the considered forms, then it is not difficult to find the angle between them. In this case it is sufficient to find the angle between the vectors which are parallel to the straight lines. For instance, if the straight lines are represented by equations in canonical form (***), then for one of the two angles θ formed by the lines we obtain

$$\cos\theta = \frac{k'k'' + l'l'' + m'm''}{\sqrt{k'^2 + l'^2 + m'^2}\,\sqrt{k''^2 + l''^2 + m''^2}}.$$

EXERCISES

1. Show that if for the straight lines specified by the equations (***),

$$\begin{vmatrix} x'-x'' & y'-y'' & z'-z'' \\ k' & l' & m' \\ k'' & l'' & m'' \end{vmatrix} = 0,$$

then the lines are either parallel, or intersect.

2. Find the distance between two skew lines represented by equations in canonical form.

3. Find the parallelism (perpendicularity) condition for the straight line

$$\left.\begin{array}{l} a_1x + b_1y + c_1z + d_1 = 0, \\ a_2x + b_2y + c_2z + d_2 = 0, \end{array}\right\}$$

and the plane

$$ax + by + cz + d = 0.$$

4. Find the parallelism (perpendicularity) condition for the straight lines

$$\left.\begin{array}{l} a_1x + b_1y + c_1z + d_1 = 0, \\ a_2x + b_2y + c_2z + d_2 = 0 \end{array}\right\}$$

and

$$\left.\begin{array}{l} a_3x + b_3y + c_3z + d_3 = 0, \\ a_4x + b_4y + c_4z + d_4 = 0. \end{array}\right\}$$

5. Find the equation of a conical surface with the vertice (x_0, y_0, z_0), whose generatrices intersect the plane

$$ax + by + cz + d = 0$$

at an angle α.

6. Write the equation of the straight line passing through the point (x_0, y_0, z_0) and parallel to the planes

$$a_1x + b_1y + c_1z + d_1 = 0,$$

$$a_2x + b_2y + c_2z + d_2 = 0.$$

7. Form the equation of a conical surface with the vertex at point $(0, 0, 2R)$ if it passes through a circle specified by the intersection of the sphere

$$x^2 + y^2 + z^2 = 2Rz$$

with the plane

$$ax + by + cz + d = 0.$$

Find out what is the intersection of this conical surface and the xy-plane.

8. *Stereographic projection of a sphere* on a plane is defined as the projection from an arbitrary point of this sphere on the tangent plane at the diametrically opposite point. Show that in stereographic projecting to the circles on the sphere there correspond circles and straight lines on the plane of projection.

Sec. 6-7. Basic Problems on the Straight Line and the Plane

Form the equation of an arbitrary plane passing through the point (x_0, y_0, z_0).

Any plane is specified by an equation of the form

$$ax + by + cz + d = 0.$$

Since the point (x_0, y_0, z_0) belongs to the plane, then

$$ax_0 + by_0 + cz_0 + d = 0.$$

Hence the equation of the required plane is

$$ax + by + cz - (ax_0 + by_0 + cz_0) = 0,$$

or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Obviously, for any a, b, c this equation is satisfied by the point (x_0, y_0, z_0).

Form the equation of an arbitrary straight line passing through the point (x_0, y_0, z_0).

The required equation is

$$\frac{x - x_0}{k} = \frac{y - y_0}{l} = \frac{z - z_0}{m}.$$

Indeed, this equation specifies a straight line passing through the point (x_0, y_0, z_0) whose coordinates obviously satisfy the equation. Taking arbitrary (not all equal to zero) values for k, l, m, we obtain a straight line of an arbitrary direction.

Form the equation of a straight line passing through two given points (x', y', z') *and* (x'', y'', z'').

The equation of the straight line may be written in the form

$$\frac{x - x'}{k} = \frac{y - y'}{l} = \frac{z - z'}{m}.$$

Since the second point lies on the line, then

$$\frac{x'' - x'}{k} = \frac{y'' - y'}{l} = \frac{z'' - z'}{m}.$$

This allows us to eliminate k, l, m, and we get the equation

$$\frac{x - x'}{x'' - x'} = \frac{y - y'}{y'' - y'} = \frac{z - z'}{z'' - z'}.$$

Form the equation of a plane passing through three points $A'(x', y', z')$, $A''(x'', y'', z'')$, $A'''(x''', y''', z''')$, *not lying on a straight line.*

Let $A(x, y, z)$ be an arbitrary point belonging to the required plane. The three vectors

$$\overrightarrow{A'A}, \quad \overrightarrow{A'A''}, \quad \overrightarrow{A'A'''}$$

lie in one plane. Consequently,

$$(\overrightarrow{A'A}, \quad \overrightarrow{A'A''}, \quad \overrightarrow{A'A'''}) = 0,$$

and we get the required equation

$$\begin{vmatrix} x-x' & y-y' & z-z' \\ x''-x' & y''-y' & z''-z' \\ x'''-x' & y'''-y' & z'''-z' \end{vmatrix}=0.$$

Form the equation of a plane passing through a given point (x_0, y_0, z_0) *and parallel to the plane*

$$ax+by+cz+d=0.$$

The required equation is

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0.$$

Indeed, this plane passes through the given point and is parallel to the given plane.

Form the equation of a straight line passing through a given point (x_0, y_0, z_0) *parallel to a given straight line*

$$\frac{x-x'}{k}=\frac{y-y'}{l}=\frac{z-z'}{m}.$$

The required equation is

$$\frac{x-x_0}{k}=\frac{y-y_0}{l}=\frac{z-z_0}{m}.$$

A straight line passing through a point (x_0, y_0, z_0) *and perpendicular to a plane*

$$ax+by+cz+d=0,$$

is specified by the equation

$$\frac{x-x_0}{a}=\frac{y-y_0}{l}=\frac{z-z_0}{c}.$$

A plane perpendicular to a straight line

$$\frac{x-x'}{k}=\frac{y-y'}{l}=\frac{z-z'}{m},$$

passing through a point (x_0, y_0, z_0), *is specified by the equation*

$$k(x-x_0)+l(y-y_0)+m(z-z_0)=0.$$

Let us form the equation of a plane passing through a point (x_0, y_0, z_0) *and parallel to the straight lines*

$$\frac{x-x'}{k'}=\frac{y-y'}{l'}=\frac{z-z'}{m'},$$
$$\frac{x-x''}{k''}=\frac{y-y''}{l''}=\frac{z-z''}{m''}.$$

Since the vectors (k', l', m') and (k'', l'', m'') are parallel to the plane, their vector product is perpendicular to the plane. Hence the required equation is

$$(x-x_0)\begin{vmatrix} l' & m' \\ l'' & m'' \end{vmatrix}+(y-y_0)\begin{vmatrix} m' & k' \\ m'' & k'' \end{vmatrix}+(z-z_0)\begin{vmatrix} k' & l' \\ k'' & l'' \end{vmatrix}=0,$$

which can be rewritten in a compact form:

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ k' & l' & m' \\ k'' & l'' & m'' \end{vmatrix}=0.$$

EXERCISES

1. Form the equation of a plane equidistant from two skew lines represented by equations in canonical form.

2. Show that any plane passing through the straight line

$$\left.\begin{aligned} a_1x+b_1y+c_1z+d_1=0, \\ a_2x+b_2y+c_2z+d_2=0, \end{aligned}\right\}$$

is specified by an equation of the form

$$\lambda\,(a_1x+b_1y+c_1z+d_1)+\mu\,(a_2x+b_2y+c_2z+d_2)=0.$$

3. Show that the plane passing through the straight line

$$\frac{x-x'}{k}=\frac{y-y'}{l}=\frac{z-z'}{m}$$

and the point (x_0, y_0, z_0), not lying on the line is specified by the equation

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ x'-x_0 & y'-y_0 & z'-z_0 \\ k & l & m \end{vmatrix} = 0.$$

4. Show that any straight line intersecting the given lines:

$$\left.\begin{aligned} a_1x+b_1y+c_1z+d_1=0, \\ a_2x+b_2y+c_2z+d_2=0. \end{aligned}\right\}$$

$$\left.\begin{aligned} a_3x+b_3y+c_3z+d_3=0, \\ a_4x+b_4y+c_4z+d_4=0, \end{aligned}\right\}$$

is represented by the equations

$$\lambda(a_1x+b_1y+c_1z+d_1)+\mu\ (a_2x+b_2y+c_2z+d_2)=0,$$
$$\lambda'\ (a_3x+b_3y+c_3z+d_3)+\mu'\ (a_4x+b_4y+c_4z+d_4)=0.$$

5. Show that the conical surface generated by straight lines passing through the origin and intersecting the curve $\varphi\ (x, y) = 0$, $z = 1$ is specified by the equation

$$\varphi\left(\frac{x}{z}, \ \frac{y}{z}\right)=0.$$

Chapter 7

Surfaces of the Second Order (Quadric Surfaces)

Sec. 7-1. A Special System of Coordinates

The *surface of the second order* (or the quadric surface) is defined as a locus of points in space whose Cartesian coordinates satisfy the equation of the form

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{13}xz + \\ + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0. \qquad (*)$$

Obviously, *this definition is invariant to the system of coordinates chosen.* Indeed, the equation of the surface in any other system of coordinates $x'y'z'$ is obtained from the equation $(*)$ by substituting x, y, and z by linear expressions with respect to x', y', z', and, consequently, in the coordinates x', y', z' will also have the form $(*)$.

Any plane intersects a quadric surface along a curve of the second order. Indeed, since the determination of surface is invariant with reference to the coordinate system chosen, we may regard the plane xy $(z = 0)$ as a secant plane. And this plane obviously intersects the surface along the second-order curve

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{14}x + 2a_{24}y + a_{44} = 0.$$

In particular, a right circular cone with the z-axis

$$\lambda z^2 = x^2 + y^2$$

is a surface of the second order and, consequently, is intersected by any plane along a second-order curve. If the secant plane does not pass through the vertex, then a pair of straight lines is excluded and we have an ellipse, hyperbola or parabola.

To study the geometrical properties of a quadric surface it is only natural to refer it to such a coordinate system in which its equation will have the simplest form.

Now *we are going to indicate a coordinate system in which the equation of our surface will become considerably simplified. Namely, the coefficients of* yz, xz, and xy *in the equation will be equal to zero.*

Consider the function $F(A)$ of a point $A(x, y, z)$ defined in the entire space, except for the origin, by the equality

$$F(A) = \frac{a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{13}xz}{x^2 + y^2 + z^2}.$$

On the unit sphere ($x^2 + y^2 + z^2 = 1$) it is bounded and, consequently, reaches the absolute minimum at some point A_0. And since it is constant along any ray emanating from the origin ($F(\lambda x, \lambda y, \lambda z) = F(x, y, z)$), then at A_0 the function F reaches the absolute minimum of values with reference to the whole space (and not only on the unit sphere).

Let us introduce new Cartesian coordinates x', y', z' with the origin O retained and taking the ray OA_0 for the positive semi-axis z. As is known, the relation between the coordinates x, y, z and x', y', z' is established by the formulas of the form

$$\left.\begin{aligned} x &= \alpha_{11}x' + \alpha_{12}y' + \alpha_{13}z', \\ y &= \alpha_{21}x' + \alpha_{22}y' + \alpha_{23}z', \\ z &= \alpha_{31}x' + \alpha_{32}y' + \alpha_{33}z'. \end{aligned}\right\} \qquad (**)$$

The equation of the surface in the new coordinates x', y', z' is obtained from the equation (*) by replacing x, y, z by x', y', z' according to the formulas (**) and has the form

$$a'_{11}x'^2 + a'_{22}y'^2 + a'_{33}z'^2 + 2a'_{12}x'y' + 2a'_{23}y'z' + 2a'_{13}x'z' + \\ + 2a'_{14}x' + 2a'_{24}y' + 2a'_{34}z' + a'_{44} = 0.$$

The function F in the new coordinates has the form

$$F(A) = \frac{a'_{11}x'^2 + a'_{22}y'^2 + a_{33}z'^2 + 2a'_{12}x'y' + 2a'_{23}y'z' + 2a'_{13}x'z'}{x'^2 + y'^2 + z'^2}$$

and is obtained by replacing x, y, z in the old expression for F by x', y', z' also according to the formulas (**). The form of the denominator remains unchanged, since it represents the square of the distance of the point A from the origin which is expressed in both systems in the same way.

According to the chosen system of coordinates $x'y'z'$ the minimum of the function F is reached at $x' = 0$, $y' = 0$, $z' = 1$. Therefore, if in the expression for F we put $x' = 0$, $z' = 1$, then we get a function of single variable

$$f(y') = \frac{a'_{22}y'^2 + 2a'_{23}y' + a'_{33}}{1 + y'^2},$$

which reaches the minimum at $y' = 0$. Consequently,

$$\frac{df(y')}{dy'} = 0 \quad \text{for} \quad y' = 0.$$

But

$$\left.\frac{df(y')}{dy'}\right|_{y'=0} = 2a'_{23}.$$

Thus, the coefficient of $y'z'$ in the equation of the surface is equal to zero. It is shown in a similar way that the coefficient of $x'z'$ is also equal to zero.

Hence, the equation of the surface in the coordinate system $x'y'z'$ will be

$$a'_{11}x'^2 + 2a'_{12}x'y' + a'_{22}y'^2 + 2a'_{14}x' + 2a'_{24}y' + 2a'_{34}z' + \\ + a'_{33}z'^2 + a'_{44} = 0.$$

If now we introduce new coordinates x'', y'', z'' according to the formulas

$$x' = x'' \cos\theta + y'' \sin\theta,$$
$$y' = -x'' \sin\theta + y'' \cos\theta,$$
$$z' = z'',$$

then, the same as in the case of the second-order curves considered in Sec. 3-8, by appropriate choice of the angle θ we can achieve the coefficient of $x''y''$ also equal to zero.

And so, *there exists such a system of rectangular Cartesian coordinates in which the equation of the surface has the form*

$$a_{11}x^2 + a_2\, y^2 + a_{33}z^2 + 2a_1x + 2a_2y + 2a_3z + a = 0. \quad (*)$$

Sec. 7-2. Quadric Surfaces Classified

As it was shown in the preceding section, by transition to an appropriate system of coordinates the equation of a quadric surface can be reduced to the form

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_1x + 2a_2y + 2a_3z + a = 0. (*)$$

We shall distinguish three basic cases:

A: all the coefficients of the squares of the coordinates in the equation (*) are non-zero;

B: two coefficients are non-zero, and the third one, for instance a_{33}, is equal to zero;

C: one coefficient, say a_{33}, is non-zero, and two others are equal to zero.

In Case A, by transition to a new coordinate system according to the formulas

$$x' = x + \frac{a_1}{a_{11}}, \quad y' = y + \frac{a_2}{a_{22}}, \quad z' = z + \frac{a_3}{a_{33}},$$

which corresponds to the translation of the origin, we reduce the equation to the form

$$\alpha x'^2 + \beta y'^2 + \gamma z'^2 + \delta = 0.$$

Here we distinguish the following subcases:

A_1: $\delta = 0$. The surface is a *cone* either *imaginary* if α, β, γ are of the same sign, or *real* if among the numbers α, β, γ there are numbers having different signs.

A_2: $\delta \neq 0$, α, β, γ are of the same sign. The surface represents an *ellipsoid* either *imaginary* if α, β, γ, δ are of the same sign, or *real* if the sign of δ is opposite to that of α, β, γ.

A_3: $\delta \neq 0$, of the four coefficients α, β, γ, δ two coefficients are of one sign, the remaining two having the opposite sign. The surface is a *hyperboloid of one sheet.*

A_4: $\delta \neq 0$, one of the first three coefficients has a sign opposite to that of the remaining coefficients. The surface is a *two-sheeted hyperboloid.*

In Case B by transition to new coordinates according to the formulas

$$x' = x + \frac{a_1}{a_{11}}, \quad y' = y + \frac{a_2}{a_{22}}, \quad z' = z$$

we reduce the equation of the surface to the form

$$\alpha x'^2 + \beta y'^2 + 2pz' + q = 0.$$

Here we shall distinguish the following subcases:

B_1: $p = 0$, $q = 0$. *The surface decomposes into a pair of planes*

$$x' \pm \sqrt{-\frac{\beta}{\alpha}}\, y' = 0$$

either *imaginary* if α and β are of the same sign, or *real* if α and β have opposite signs.

B_2: $p = 0$, $q \neq 0$. The surface represents a *cylinder* either *imaginary* if α, β, and q are of the same sign, or *real* if there are coefficients with different signs. In particular, if α and β are of the same sign, then we have an *elliptic cylinder*, and if α and β have different signs, then we have a *hyperbolic cylinder*.

B_3: $p \neq 0$. *Paraboloids.* Passing over to new coordinates

$$x'' = x', \quad y'' = y', \quad z'' = z' + \frac{q}{2p},$$

we reduce the equation of the surface to the form

$$\alpha x''^2 + \beta y''^2 + 2pz'' = 0.$$

The paraboloid is *elliptic* if α and β are of the same sign, and *hyperbolic* if α and β are of different signs.

In Case C we pass over to new coordinates x', y', z':

$$x' = x, \quad y' = y, \quad z' = z + \frac{a_3}{a_{33}}.$$

Then the equation will take the form

$$\gamma z'^2 + px + qy + r = 0$$

and we may distinguish the following subcases:

C_1: $p = 0$, $q = 0$. *The surface decomposes into a pair of parallel planes*: *imaginary* if γ and r are of the same sign, or *real* if γ and r have opposite signs, or *coincident* if $r = 0$.

C_2: at least one of the coefficients p or q is non-zero. Preserving the direction of the z-axis, we take the plane

$px + qy + r = 0$ for the plane $y'z'$. Then the equation will take the form

$$\gamma z'^2 + \delta x' = 0.$$

The surface is a *parabolic cylinder*.

EXERCISES

1. The curve in the xy-plane

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a = 0$$

represents an ellipse (hyperbola, parabola). What does the quadric surface represent

$$z = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a?$$

2. Show that the quadric surface

$$\lambda\,(a_1x + b_1y + c_1z + d_1)^2 + \mu\,(a_2x + b_2y + c_2z + d_2)^2 = 0$$

decomposes into a pair of planes.

3. To obtain the projection (on the xy-plane) of the curve of intersection of the surface

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + \ldots + a_{44} = 0 \qquad (*)$$

with the plane

$$z = ax + by + c,$$

one has to substitute $z = ax + by + c$ in the equation (*). Show this.

4. Show that the sections of a quadric surface by parallel planes are homothetic.

5. Show that the conical surface generated by straight lines passing through a given point and intersecting a second-order curve is a quadric surface.

6. Let

$$f(x, y, z) = 0, \quad \varphi(x, y, z) = 0$$

be equations of two quadric surfaces. Show that the equation of the quadric surface passing through the point (x_0, y_0, z_0) and the intersection of two given surfaces will be

$$f(x, y, z)\,\varphi(x_0, y_0, z_0) - \varphi(x, y, z)\,f(x_0, y_0, z_0) = 0.$$

7. Show that the straight line specified by the equations

$$\left.\begin{aligned}(a_1x+b_1y+c_1z+d_1)+\lambda(\alpha_1x+\beta_1y+\gamma_1z+\delta_1)=0,\\(a_2x+b_2y+c_2z+d_2)+\frac{1}{\lambda}(\alpha_2x+\beta_2y+\gamma_2z+\delta_2)=0,\end{aligned}\right\}$$

lies entirely on the quadric surface

$$(a_1x+b_1y+c_1z+d_1)(a_2x+b_2y+c_2z+d_2)-$$
$$-(\alpha_1x+\beta_1y+\gamma_1z+\delta_1)(\alpha_2x+\beta_2y+\gamma_2z+\delta_2)=0.$$

8. Find out what is the surface generated by straight lines intersecting three given straight lines which are non-parallel and do not intersect.

9. Form the equation of the surface generated by the straight line

$$\left.\begin{aligned}z=ax+b,\\z=cy+d\end{aligned}\right\}\quad(a,\ b,\ c,\ d\neq 0)$$

rotating about the z-axis.

Sec. 7-3. The Ellipsoid

The equation of the ellipsoid is (Fig. 74)

$$\alpha x^2+\beta y^2+\gamma z^2+\delta=0.$$

Dividing it by δ and putting $\delta/\alpha=-a^2$, $\delta/\beta=-b^2$, $\delta/\gamma=-c^2$, we reduce it to the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \qquad (*)$$

where a, b, c are the *semi-axes* of the ellipsoid.

It is seen from the equation (*) that the coordinate planes are the planes of symmetry of the ellipsoid, and the origin is the centre of symmetry.

Like the ellipse which is obtained from the circle by uniform compression, any ellipsoid is generated by uniformly compressing a sphere with respect to two mutually perpendicular planes. Namely, if a is the greatest semi-axis of the ellipsoid, then it can be obtained from the sphere

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{a^2}-1=0$$

by uniformly compressing it with respect to the xy-plane with the compression ratio c/a and with respect to the xz-plane with the compression ratio b/a.

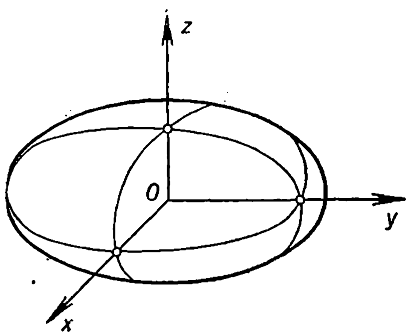


Fig. 74.

If two semi-axes of an ellipsoid are equal, for instance, $a = b$, then it is called an *ellipsoid of revolution.*

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2}-1=0.$$

Intersecting it with any plane $z = h$ parallel to the xy-plane, we obtain a circle

$$x^2+y^2=\left(1-\frac{h^2}{c^2}\right)a^2, \quad z=h$$

with centre on the z-axis. Hence, in this case the *ellipsoid is generated by revolving the ellipse*

$$\frac{x^2}{a^2}+\frac{z^2}{c^2}-1=0,$$

contained in the xz-plane about the z-axis (Fig. 75).

If all the three semi-axes are equal, then it represents a sphere.

The line of intersection of an ellipsoid with an arbitrary plane is an ellipse.

Indeed, this line represents a curve of the second order. Since this line is finite (the ellipsoid is a finite figure), it cannot be a hyperbola, or a parabola. Nor can it be a pair of straight lines, and consequently it is an ellipse.

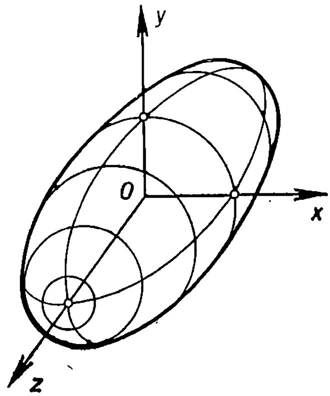


Fig. 75.

EXERCISES

1. If $a < c$, then the ellipsoid of revolution

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1,$$

represents a locus of points the sum of whose distances from two given points (the foci) are constant. Find the foci of the ellipsoid.

2. Suppose we have an ellipsoid

$$\alpha x^2 + \beta y^2 + \gamma z^2 + \delta = 0.$$

Show that if the surface

$$\alpha x^2 + \beta y^2 + \gamma z^2 + \delta - \lambda (x^2 + y^2 + z^2 + \mu) = 0$$

decomposes into a pair of planes, then these planes intersect the ellipsoid along circles. Use this fact to justify the method of finding circular sections of the ellipsoid.

3. Where are the points in space situated for which

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 < 0?$$

4. Show that the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

may be specified by the equations in parametric form:

$$x = a \cos u \cos v, \quad y = b \cos u \sin v, \quad z = c \sin u.$$

5. What is the surface

$$(a_1x + b_1y + c_1z)^2 + (a_2x + b_2y + c_2z)^2 + (a_3x + b_3y + c_3z)^2 = 1,$$

if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0?$$

Sec. 7-4. Hyperboloids

Like the case of the ellipsoid, the equation of hyperboloids can be reduced to the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0$$

(a hyperboloid of one sheet, Fig. 76),

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0$$

(a hyperboloid of two sheets, Fig. 77).

In both hyperboloids the coordinate planes serve as the *planes of symmetry*, and the origin of coordinates as the *centre of symmetry*.

If the semi-axes a and b of the hyperboloid are equal, then it is called a *hyperboloid of revolution* and is obtained by revolving (about the z-axis) the hyperbola

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} - 1 = 0, \quad y = 0$$

in the case of a hyperboloid of one sheet and the hyperbola

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} + 1 = 0, \quad y = 0$$

in the case of a hyperboloid of two sheets.

A general-type hyperboloid ($a \neq b$) can be obtained from a hyperboloid of revolution ($a = b$) by uniformly compressing (or stretching) the latter with respect to the xz-plane in the ratio b/a.

Hyperboloids are cut by an arbitrary plane in various conic sections. For instance, the planes $z = h$ parallel

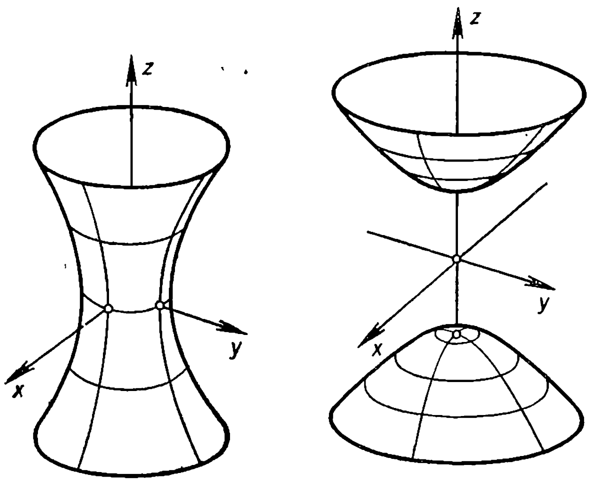


Fig. 76. *Fig. 77.*

to the xy-plane cut a hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0$$

in ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{h^2}{c^2} - 1 = 0, \quad z = h,$$

and the planes $y = h$ ($|h| \neq b$) parallel to the xz-plane in hyperbolas

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} - 1 + \frac{h^2}{b^2} = 0, \quad y = h.$$

The plane $y = b$ intersects the hyperboloid along two straight lines:

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 0, \quad y = b.$$

EXERCISES

1. Find the circular sections of the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0.$$

2. Show that through any point in space not belonging to the coordinate planes, there pass three surfaces of the family

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1$$

(λ, the parameter): an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets.

Sec. 7-5. Paraboloids

The equations of paraboloids are reduced to the form

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

(an elliptic paraboloid, Fig. 78),

$$z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

(a hyperbolic paraboloid, Fig. 79).

The xz- and yz-planes are the planes of symmetry of paraboloids. Their intersection (the z-axis) is called the *axis of a paraboloid*, and the intersection of its axis with the surface is termed the *vertex*.

If $a = b$ an elliptic paraboloid is said to be a *paraboloid of revolution*. It is formed by revolving a parabola

$$z=\frac{x^2}{a^2}, \quad y=0$$

about the z-axis. This is the special case of the elliptic paraboloid in which the cross-sections perpendicular to the axis are circles.

A general-type elliptic paraboloid can be obtained from a paraboloid of revolution

$$z=\frac{x^2}{a^2}+\frac{y^2}{a^2}$$

by uniformly compressing (stretching) it with respect to the xz-plane.

Both paraboloids (elliptic and hyperbolic) are cut by planes parallel to the xz- and yz-planes in parabolas that

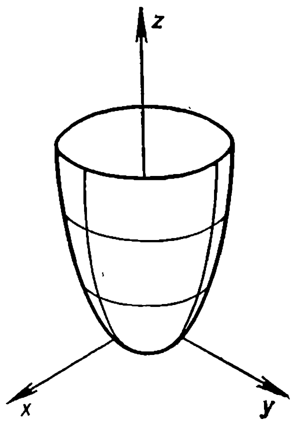


Fig. 78.

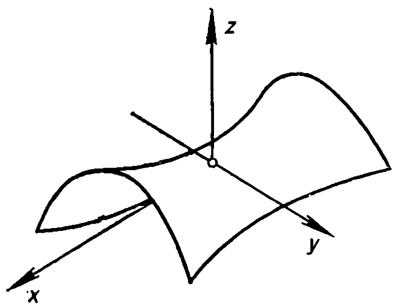


Fig. 79.

are parallel and equal. Indeed, the planes $x = h$ cut an elliptic paraboloid in parabolas

$$z - \frac{h^2}{a^2} = \frac{y^2}{b^2}, \quad x = h.$$

If each of these parabolas is displaced in the direction of z, by a line segment h^2/a^2, then we obtain one and the same parabola

$$z = \frac{y^2}{b^2}, \quad x = h.$$

Whence it follows that an *elliptic paraboloid is generated by translating a parabola* $z = \frac{y^2}{b^2}$, $x = 0$, *with its vertex moving along a parabola* $z = \frac{x^2}{a^2}$, $y = 0$ (Fig. 80).

A hyperbolic paraboloid is generated in a similar way (Fig. 81).

The planes parallel to the xy-plane, except for this plane itself, cut an elliptic paraboloid in ellipses, and

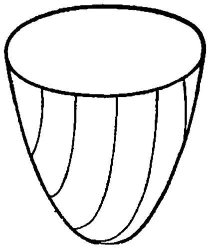


Fig. 80.

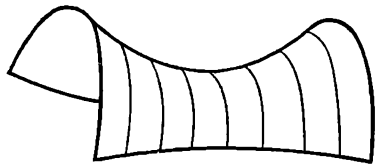


Fig. 81.

a hyperbolic paraboloid in hyperbolas. The xy-plane intersects a hyperbolic paraboloid along two straight lines.

EXERCISES

1. Show that an elliptic paraboloid of revolution represents a locus of points equidistant from a plane and a point (the focus). Find the focus of the elliptic paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{a^2}.$$

2. Show that no plane cuts an elliptic paraboloid in hyperbolas and a hyperbolic paraboloid in ellipses.

Sec. 7-6. The Cone and Cylinders

The equation of the cone and cylinders of the second order may be written in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \quad \text{(a cone, Fig. 82)},$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \quad \text{(an elliptic cylinder, Fig. 83)},$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0 \quad \text{(a hyperbolic cylinder, Fig. 84)},$$

$$\frac{x^2}{a^2} - py = 0 \quad \text{(a parabolic cylinder, Fig. 85)}.$$

An arbitrary cone is obtained from a circular cone

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{c^2}=0$$

by compressing (stretching) it uniformly with respect to the xz-plane.

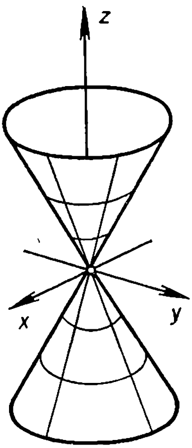


Fig. 82.

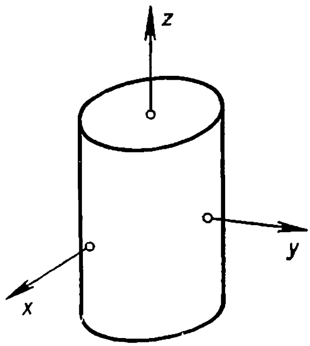


Fig. 83.

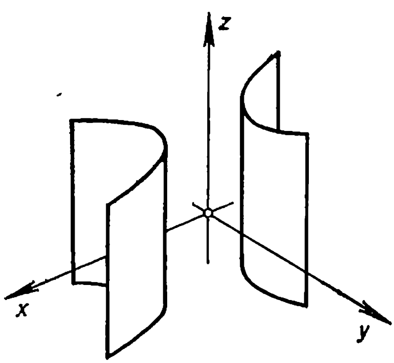


Fig. 84.

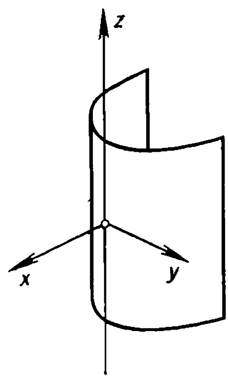


Fig. 85

Elliptic, hyperbolic, and parabolic cylinders intersect the xy-plane along an ellipse, hyperbola, and parabola, respectively, and are generated by straight lines parallel to the z-axis which intersect the mentioned curves.

An arbitrary elliptic cylinder is obtained from a circular cylinder by compressing (stretching) the latter uniformly with respect to the xz-plane.

We conclude with a note that the cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0,$$

which is called the *asymptotic cone*, is related with the hyperboloids of one and two sheets

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}\pm 1=0$$

in a natural way.

Any plane passing through the z-axis cuts the hyperboloids in hyperbolas, and the cone along two elements which are the asymptotes of these hyperbolas. In particular, the xz-plane ($y=0$) cuts the hyperboloids in hyperbolas

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}\pm 1=0,$$

and the cone along two straight lines

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=0,$$

which are the asymptotes of these hyperbolas.

EXERCISES

1. Show that the equation of a circular cone with the vertex at the origin, the axis $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$, and the vertex angle 2α may be written in the form

$$\frac{(\lambda x+\mu y+\nu z)^2}{(x^2+y^2+z^2)(\lambda^2+\mu^2+\nu^2)}=(\cos\alpha)^2.$$

2. Show that the equation of a circular cylinder of radius R and with the axis $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$ may be written in the form

$$x^2+y^2+z^2-R^2=\frac{(\lambda x+\mu y+\nu z)^2}{\lambda^2+\mu^2+\nu^2}.$$

Sec. 7-7. Rectilinear Generators on Quadric Surfaces

Cones and cylinders are not the only surfaces of the second order containing rectilinear generators. A hyperboloid of one sheet and a hyperbolic paraboloid turn out to possess this property as well.

Indeed, *any straight line* g_λ, *specified by the equations*

$$z=\lambda\left(\frac{x}{a}+\frac{y}{b}\right),\quad 1=\frac{1}{\lambda}\left(\frac{x}{a}-\frac{y}{b}\right),\tag{*}$$

lies on the hyperbolic paraboloid

$$z=\frac{x^2}{a^2}-\frac{y^2}{b^2}\tag{**}$$

since any point (x, y, z) satisfying the equation (*) also satisfies the equation (**) which is obtained from them as a corollary by termwise multiplication.

In addition to the family g_λ, *one more family of straight lines* g'_λ *is located on a hyperbolic paraboloid*:

$$z=\lambda\left(\frac{x}{a}-\frac{y}{b}\right),\quad 1=\frac{1}{\lambda}\left(\frac{x}{a}+\frac{y}{b}\right).$$

Analogously: *on the hyperboloid of one sheet*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0$$

there are two families of rectilinear generators

$$g_\lambda:\quad \frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right),\quad \frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}\left(1+\frac{y}{b}\right);$$

$$g'_\lambda:\quad \frac{x}{a}-\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right),\quad \frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right).$$

In both cases (a hyperbolic paraboloid and hyperboloid of one sheet) *rectilinear generators belonging to one family do not intersect, whereas those belonging to different families intersect.*

The presence of rectilinear generators on the surfaces of a hyperbolic paraboloid and a hyperboloid of one sheet makes it possible to introduce a new method of generating these surfaces. Namely, let us take three rectilinear generators g_1, g_2, g_3 belonging to one family.

Then each rectilinear generator g belonging to the second family intersects g_1, g_2, g_3. Consequently, the surface is generated by the straight lines g which intersect the three given lines (Fig. 86).

As to the hyperboloid of revolution of one sheet, it is formed also by revolving any of its rectilinear generators about the axis of the ruled surface (Fig. 87).

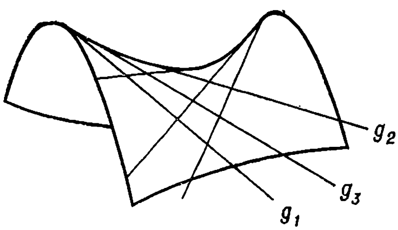


Fig. 86.

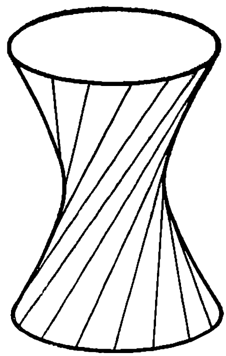


Fig. 87.

We conclude with a note that there are rectilinear generators on other quadric surfaces, but only imaginary. For instance, on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$$

there are located two families of imaginary straight lines:

$$g_\lambda: \quad \frac{x}{a}+i\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right), \quad \frac{x}{a}-i\frac{z}{c}=\frac{1}{\lambda}\left(1+\frac{y}{b}\right);$$

$$g'_\lambda: \quad \frac{x}{a}+i\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right), \quad \frac{x}{a}-i\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right).$$

EXERCISES

1. Show that the plane $\frac{xx_0}{a^2}-\frac{yy_0}{b^2}+\frac{z+z_0}{2}=0$ passing through the point (x_0, y_0, z_0) of the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}+z=0$ intersects the paraboloid

along two rectilinear generators belonging to two different families.

2. Find the rectilinear generators of the hyperbolic paraboloid $z = axy$.

3. Form the equation of a surface generated by straight lines parallel to the xy-plane and intersecting two given skew lines.

Sec. 7-8. Diameters and Diametral Planes of a Quadric Surface

A straight line, as a rule, intersects a quadric surface at two points. If there are two points of intersection, then the line segment with the end-points at the points of intersection is called the *chord*.

The mid-points of parallel chords of a quadric surface lie in a plane (termed the *diametral plane*).

Let us prove this. As it was shown in Sec. 7-1, there exists a system of coordinates in which the equation of the surface has the form

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_1x + 2a_2y + 2a_3z + a = 0. \quad (*)$$

Let the chords be parallel to the line $\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}$, and let $\bar{x}, \bar{y}, \bar{z}$ denote the coordinates of a mid-point of an arbitrary chord. Then the coordinates of the end-points of the chord may be written in the form $x = \bar{x} + \lambda t$, $y = \bar{y} + \mu t$, $z = \bar{z} + \nu t$ for one end, and $x = \bar{x} - \lambda t$, $y = \bar{y} - \mu t$, $z = \bar{z} - \nu t$ for the other.

Since the end-points of the chord belong to the surface, their coordinates satisfy the equation (*). Whence

$$a_{11}\bar{x}^2 + a_{22}\bar{y}^2 + a_{33}\bar{z}^2 + 2a_1\bar{x} + 2a_2\bar{y} + 2a_3\bar{z} + a + \\ + 2t\,(\lambda a_{11}\bar{x} + \mu a_{22}\bar{y} + \nu a_{33}\bar{z} + \lambda a_1 + \mu a_2 + \nu a_3) + \\ + t^2\,(a_{11}\lambda^2 + a_{22}\mu^2 + a_{33}\nu^2) = 0.$$

Since this equality holds irrespective of the sign taken for t, the coefficient of t is equal to zero:

$$\lambda\,(a_{11}\bar{x} + a_1) + \mu\,(a_{22}\bar{y} + a^2) + \nu\,(a_{33}\bar{z} + a_3) = 0. \quad (**)$$

Thus, the coordinates of the mid-points of chords satisfy the equation of the plane which was required to be proved.

Obviously, *if a surface has a centre, then the diametral plane passes through the centre.*

In the case of the paraboloid ($a_{33} = 0$) *all diametral planes are parallel to its axis (to the z-axis).*

An elliptic (hyperbolic) cylinder has an infinite number of centres situated on its axis. Therefore, each diametral plane of the cylinder passes through its axis. This circumstance is reflected in the equation of diametral planes. In the case of a parabolic cylinder all diametral planes are parallel.

The diametral planes of the cone pass through its vertex.

Diametral planes possess the following general property: *the diametral planes corresponding to the chords parallel to the plane α either intersect along a straight line g, or are parallel. The diametral plane corresponding to the chords parallel to g is parallel to α.*

Let us prove this. Let e (λ, μ, ν) and e' (λ', μ', ν') be non-zero, non-parallel vectors in the plane α. Then any vector contained in this plane may be represented in the form e_ξ $(\xi\lambda + \xi'\lambda',\ \xi\mu + \xi'\mu',\ \xi\nu + \xi'\nu')$. The diametral plane corresponding to the chords parallel to the vector e_ξ will be

$$\xi \{\lambda (a_{11}x + a_1) + \mu (a_{22}y + a_2) + \nu (a_{33}z + a_3)\} + \\ + \xi' \{\lambda' (a_{11}x + a_1) + \mu' (a_{22}y + a_2) + \nu' (a_{33}z + a_3)\} = 0$$

and, consequently, for any ξ, ξ' passes through the line of intersection of the planes

$$\left.\begin{aligned} \lambda (a_{11}x + a_1) + \mu (a_{22}y + a_2) + \nu (a_{33}z + a_3) = 0, \\ \lambda' (a_{11}x + a_1) + \mu' (a_{22}y + a_2) + \nu' (a_{33}z + a_3) = 0, \end{aligned}\right\} \quad (***)$$

if they intersect, and is parallel to them if the planes are parallel. Suppose the planes (***) intersect and $(\lambda'', \mu'', \nu'')$ is a vector parallel to the line of intersection. Then

$$\left.\begin{aligned} \lambda''\lambda a_{11} + \mu''\mu a_{22} + \nu''\nu a_{33} = 0, \\ \lambda''\lambda' a_{11} + \mu''\mu' a_{22} + \nu''\nu' a_{33} = 0 \end{aligned}\right\} \qquad (****)$$

(parallelism of the vector $(\lambda'', \mu'', \nu'')$ to the planes (***)).

The diametral plane corresponding to the chords parallel to the vector $(\lambda'', \mu'', \nu'')$ will be

$$\lambda'' (a_{11}x + a_1) + \mu'' (a_{22}y + a_2) + \nu'' (a_{33}z + a_3) = 0.$$

From the conditions (****) it follows that this plane is parallel to the vector $e\ (\lambda, \mu, \nu)$, $e'\ (\lambda', \mu', \nu')$ and, consequently, is parallel to the plane α which contains them.

Chapter 8

Investigating Quadric Curves and Surfaces Specified by Equations of the General Form

Sec. 8-1. Transformation of the Quadratic Form to New Variables

The *quadratic form* of variables $x_1, x_2, \ldots, x_n$ is defined as a homogeneous polynomial of the second degree with respect to these variables

$$\sum_{i,j} a_{ij}x_i x_j \quad (a_{ij} = a_{ji}).$$

The *discriminant* of a quadratic form is defined as a determinant formed from its coefficients:

$$D = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}.$$

Let us replace the variables in the quadratic form according to the formulas

$$\begin{aligned} x_1 &= \alpha_{11}x_1' + \alpha_{12}x_2' + \ldots + \alpha_{1n}x_n', \\ x_2 &= \alpha_{21}x_1' + \alpha_{22}x_2' + \ldots + \alpha_{2n}x_n', \\ &\cdots\cdots\cdots\cdots\cdots \\ x_n &= \alpha_{n1}x_1' + \alpha_{n2}x_2' + \ldots + \alpha_{nn}x_n'. \end{aligned}$$

This yields a quadratic form with respect to the variables x_i'. Namely:

$$\begin{aligned} \sum_{i,j} a_{ij}x_i x_j &= \sum_{i,j} a_{ij} \left(\sum_k \alpha_{ik}x_k'\right) \left(\sum_l \alpha_{jl}x_l'\right) = \\ &= \sum_{k,l} \left(\sum_{i,j} a_{ij}\alpha_{ik}\alpha_{jl}\right) x_k' x_l' = \sum_{k,l} a_{kl}' x_k' x_l', \end{aligned}$$

where

$$a'_{kl} = \sum_{i,j} a_{ij}\alpha_{ik}\alpha_{jl}.$$

Let us find out what the discriminant D' of the obtained form equals to. Put

$$\sum_{i} a_{ij}\alpha_{ik} = b_{jk}. \qquad (*)$$

Then

$$a'_{kl} = \sum_{j} b_{jk}\alpha_{jl}$$

and, consequently,

$$D' = \begin{vmatrix} a'_{11} & \dots & a'_{1n} \\ \cdot & \cdot & \cdot \\ a'_{n1} & \dots & a'_{nn} \end{vmatrix} = \begin{vmatrix} b_{11} & \dots & b_{1n} \\ \cdot & \cdot & \cdot \\ b_{n1} & \dots & b_{nn} \end{vmatrix} \begin{vmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \cdot & \cdot & \cdot \\ \alpha_{n1} & \dots & \alpha_{nn} \end{vmatrix}.$$

But according to the formulas (*)

$$\begin{vmatrix} b_{11} & \dots & b_{1n} \\ \cdot & \cdot & \cdot \\ b_{n1} & \dots & b_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1} & \dots & a_{nn} \end{vmatrix} \begin{vmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \cdot & \cdot & \cdot \\ \alpha_{n1} & \dots & \alpha_{nn} \end{vmatrix}.$$

Thus,

$$D' = D \begin{vmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \cdot & \cdot & \cdot \\ \alpha_{n1} & \dots & \alpha_{nn} \end{vmatrix}^2,$$

i.e. *the discriminant of the transformed form is equal to the discriminant of the initial form multiplied by the square of the determinant of transformation coefficients.*

EXERCISES

1. Show that the discriminant of the quadratic form

$$(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4)(b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4)$$

is equal to zero.

2. Compute the discriminant of the quadratic form of the variables x_1, x_2, x_3, x_4:

$$\Big(\sum_i a_ix_i\Big)^2 + \Big(\sum_i b_ix_i\Big)^2 + \Big(\sum_i c_ix_i\Big)^2 + \Big(\sum_i d_ix_i\Big)^2.$$

Sec. 8-2. Invariants of the Equations of Quadric Curves and Surfaces with Reference to Transformation of Coordinates

Suppose we have an equation of a quadric surface

$$a_{11}x^2 + 2a_{12}xy + \ldots + a_{44} = 0 \qquad (*)$$

in a system of rectangular Cartesian coordinates. The equation of this surface in any other system of rectangular Cartesian coordinates $x'y'z'$ is obtained from the equation (*) if instead of x, y, z we substitute their expressions in terms of x', y', z' according to the formulas introduced in Sec. 5-4:

$$\begin{aligned} x &= \alpha_{11}x' + \alpha_{12}y' + \alpha_{13}z' + \alpha_1, \\ y &= \alpha_{21}x' + \alpha_{22}y' + \alpha_{23}z' + \alpha_2, \\ z &= \alpha_{31}x' + \alpha_{32}y' + \alpha_{33}z' + \alpha_3. \end{aligned}$$

The equation of the surface in the new coordinate system will be

$$a'_{11}x'^2 + 2a'_{12}x'y' + \ldots + a'_{44} = 0.$$

The function $\varphi(a_{11}, a_{12}, \ldots, a_{44})$, which is not a constant is called the *invariant of the equation of the surface* with reference to the transformation of the coordinates if its values are independent of the coordinate system to which the surface is referred, i.e. if

$$\varphi(a_{11}, a_{12}, \ldots, a_{44}) = \varphi(a'_{11}, a'_{12}, \ldots, a'_{44}),$$

whatever the system of coordinates $x'y'z'$ is.

Now we are going to find one of the basic invariants of the equation of the surface.

Along with transition to the new coordinate system $x'y'z'$, we shall consider the transformation of the quadratic form

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1 - \\ - \lambda(x_1^2 + x_2^2 + x_3^2)$$

to new variables x'_1, x'_2, x'_3 according to the formulas

$$\left.\begin{aligned} x_1 &= \alpha_{11}x'_1 + \alpha_{12}x'_2 + \alpha_{13}x'_3, \\ x_2 &= \alpha_{21}x'_1 + \alpha_{22}x'_2 + \alpha_{23}x'_3, \\ x_3 &= \alpha_{31}x'_1 + \alpha_{32}x'_2 + \alpha_{33}x'_3. \end{aligned}\right\} \qquad (**)$$

As a result of this transformation the first part of the quadratic form, up to the term $\lambda(x_1^2 + x_2^2 + x_3^2)$, will take the form

$$a'_{11}x_1'^2 + a'_{22}x_2'^2 + a'_{33}x_3'^2 + 2a'_{12}x'_1x'_2 + 2a'_{23}x'_2x'_3 + 2a'_{31}x'_3x'_1,$$

and the coefficients a'_{ij} will be the same as in the equation of the surface after the transition to the coordinate system x', y', z'. As far as the term $\lambda(x_1^2 + x_2^2 + x_3^2)$ is concerned, it will be transformed into $\lambda(x_1'^2 + x_2'^2 + x_3'^2)$ by virtue of the orthogonality conditions which are satisfied by the coefficients α_{ij} (see Sec. 5-4).

Since the determinant of transformation coefficients (**) is equal to ± 1, the discriminants of the quadratic forms before and after the transformation are equal. Consequently,

$$I(\lambda) = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$

is an invariant of the equation of the surface for any λ.

The determinant $I(\lambda)$ represents a polynomial with respect to λ:

$$I(\lambda) = -\lambda^3 + \lambda^2 I_1 - \lambda I_2 + I_3,$$

where

$$I_1 = a_{11} + a_{22} + a_{33},$$

$$I_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{33} & a_{31} \\ a_{13} & a_{11} \end{vmatrix},$$

$$I_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Since for two different coordinate systems xyz and $x'y'z'$

$$-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = -\lambda^3 + I'_1\lambda^2 - I'_2\lambda + I'_3$$

for all λ, then $I_1 = I'_1$, $I_2 = I'_2$, $I_3 = I'_3$ and, consequently, I_1, I_2, I_3 *are invariants of the equation of the surface.*

Let us now show that

$$I_4 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

is also an invariant.

The determinant I_4 represents the discriminant of the quadratic form

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + \ldots + a_{44}x_4^2.$$

Let us pass over to new variables x_i' using the formulas

$$\left.\begin{aligned} x_1 &= \alpha_{11}x_1' + \alpha_{12}x_2' + \alpha_{13}x_3' + \alpha_{14}x_4', \\ x_2 &= \alpha_{21}x_1' + \alpha_{22}x_2' + \alpha_{23}x_3' + \alpha_{24}x_4'. \\ x_3 &= \alpha_{31}x_1' + \alpha_{32}x_2' + \alpha_{33}x_3' + \alpha_{34}x_4', \\ x_4 &= 0 \cdot x_1' + 0 \cdot x_2' + 0 \cdot x_3' + 1 \cdot x_4'. \end{aligned}\right\} \qquad (***)$$

As a result we obtain the form

$$a_{11}'x_1'^2 + 2a_{12}'x_1'x_2' + \ldots + a_{44}'x_4'^2,$$

where a_{ij}' are the same as in the transformed equation of the surface.

Since the determinant of transformation coefficients (***) equal to the determinant of transformation coefficients (**) is equal to ± 1, the discriminants of the initial and transformed forms are equal, i.e.

$$\begin{vmatrix} a_{11} & \ldots & a_{14} \\ \cdot & \cdot & \cdot \\ a_{41} & \ldots & a_{44} \end{vmatrix} = \begin{vmatrix} a_{11}' & \ldots & a_{14}' \\ \cdot & \cdot & \cdot \\ a_{41}' & \ldots & a_{44}' \end{vmatrix},$$

and the determinant I_4 is really an invariant of the equation of the surface.

Reasoning just in the same way, *we obtain the invariants*

$$I(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix}, \quad I_1 = a_{11} + a_{22},$$

$$I_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \qquad I_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

for the equation of a quadric curve

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0$$

with reference to the transformation of coordinates.

EXERCISES

1. Compute the invariants of the equation of a surface

$$ax^2 + 2bxy + cy^2 + 2\alpha x + 2\beta y + 2\gamma z + \delta = 0.$$

2. Compute the invariants of the equation of a surface

$$x^2 + y^2 + z^2 - k^2 (ax + by + cz)^2 = 0.$$

Sec. 8-3. Investigating a Quadric Curve by Its Equation in Arbitrary Coordinates

Let there be given a quadric curve in arbitrary Cartesian coordinates xyz:

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0.$$

As we showed in Sec. 3-8, by transition to a new coordinate system, the equation of a curve may be reduced to the form

$$\alpha x^2 + \beta y^2 + ax + by + c = 0.$$

Without finding the coordinate system itself, we can simply find the coefficients α and β by means of the invariant $I(\lambda)$. Indeed,

$$\begin{vmatrix} \alpha-\lambda & 0 \\ 0 & \beta-\lambda \end{vmatrix} = \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = I(\lambda).$$

Whence it is seen that α and β are the roots of the equation $I(\lambda) = 0$, i.e. of the equation

$$\lambda^2 - I_1\lambda + I_2 = 0.$$

Suppose both roots are non-zero (it will happen if $I_2 \neq 0$). Then, as it was shown in the same Sec. 3-8, the equation of the curve can be reduced to the form

$$\alpha x^2 + \beta y^2 + \gamma = 0$$

by translating the coordinate system.

It is not difficult to find the coefficient γ using the invariant I_3. We have

$$\begin{vmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{vmatrix} = I_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Whence

$$\gamma = \frac{I_3}{\alpha\beta} = \frac{I_3}{I_2}.$$

Thus, *if* $I_2 \neq 0$, *then the equation of the curve in an appropriate system of coordinates will take the form*

$$\lambda_1 x^2 + \lambda_2 y^2 + \frac{I_3}{I_2} = 0,$$

where λ_1 *and* λ_2 *are the roots of the equation*

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0.$$

Let us now assume that one of the roots of the equation $I(\lambda) = 0$ is equal to zero (it will happen if $I_2 = 0$). Then one of the coefficients either α, or β is equal to zero; for definiteness let $\alpha = 0$. In this case (see Sec. 3-8) the curve is specified either by

$$\beta y^2 + 2\gamma x = 0$$

or by

$$\beta y^2 + \delta = 0,$$

namely by the first equation if $I_3 \neq 0$, and by the second equation if $I_3 = 0$.

Let $I_3 \neq 0$ and, consequently, the curve is specified by the equation

$$\beta y^2 + 2\gamma x = 0.$$

From the equation

$$\lambda^2 - I_1\lambda + I_2 = 0$$

with $I_2 = 0$ we find $\beta = I_1$; γ is found using the invariant I_3. Namely:

$$\begin{vmatrix} 0 & 0 & \gamma \\ 0 & \beta & 0 \\ \gamma & 0 & 0 \end{vmatrix} = I_3.$$

Whence

$$\gamma=\sqrt{-\frac{I_3}{\beta}}=\sqrt{-\frac{I_3}{I_1}}.$$

Thus, if $I_2 = 0$, $I_3 \neq 0$ the curve in the corresponding coordinates is specified by the equation

$$I_1y^2+2x\sqrt{-\frac{I_3}{I_1}}=0.$$

Let us finally consider the case when $I_2 = I_3 = 0$. We change the coefficients of the equation by small quantities ε_{ij}. We may deal with ε_{ij} in such a way that I_2 will become non-zero and the equation of the curve can be reduced to the form

$$\lambda_1x^2+\lambda_2y^2+\frac{I_3}{I_2}=0. \qquad (*)$$

And now let us proceed to the limit as ε_{ij} tend to zero. Then the equation (*) will turn into the canonical equation of the original curve.

Example. Let $I_2 = 0$, $I_3 = 0$, $a_{22} \neq 0$. We put $\varepsilon_{11} = t$, and all the remaining ε_{ij} equal to zero. Then, proceeding to the limit in the equation (*), we get

$$I_1x^2+\frac{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}{c_{22}}=0.$$

We conclude with the following note: *the vanishing of the invariant I_3 is a necessary and sufficient condition for decomposing a quadric curve into a pair of straight lines.* To be convinced of this it is sufficient to compute I_3 for the canonical forms of equations of curves.

EXERCISES

1. What condition must be satisfied by λ for the quadric curve

$$(a_{11}x^2+2a_{12}xy+\ldots+a_{33})+ \\ +\lambda(b_{11}x^2+2b_{12}xy+\ldots+b_{33})=0$$

to decompose into a pair of straight lines? Show that the lines into which this curve decomposes pass through the points of intersection of the curves

$$a_{11}x^2 + 2a_{12}xy + \ldots + a_{33} = 0,$$
$$b_{11}x^2 + 2b_{12}xy + \ldots + b_{33} = 0.$$

2. The biquadratic equation

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$$

is equivalent to the system

$$a_0y^2 + a_1xy + a_2x^2 + a_3x + a_4 = 0, \quad y - x^2 = 0.$$

Reduce the solution of the biquadratic equation to solving a cubic and a quadratic equations (see Exercise 1).

3. The equation of a hyperbola referred to the centre and one of its asymptotes has the form

$$y = \alpha x + \frac{\beta}{x}.$$

Express α and β in terms of the coefficients of the equation of this hyperbola in arbitrary coordinates.

4. If equal, mutually perpendicular diameters of an ellipse are taken for the coordinate axes, then its equation will take the form

$$x^2 + y^2 + 2\alpha xy + \delta = 0.$$

Find α and δ given the equation of the ellipse in arbitrary coordinates.

Sec. 8-4. Investigating a Quadric Surface Specified by an Equation in Arbitrary Coordinates

Let a quadric surface be specified by an equation in an arbitrary system of rectangular coordinates xyz:

$$a_{11}x^2 + 2a_{12}xy + \ldots + a_{44} = 0.$$

As is shown in Sec. 7-1, by transition to a new system of coordinates, the equation of a surface can be reduced to the form

$$\alpha x^2 + \beta y^2 + \gamma z^2 + ax + by + cz + d = 0.$$

Using the invariant $I(\lambda)$, we get

$$I(\lambda)=\begin{vmatrix}\alpha-\lambda & 0 & 0\\ 0 & \beta-\lambda & 0\\ 0 & 0 & \gamma-\lambda\end{vmatrix}=-\lambda^3+I_1\lambda^2-I_2\lambda+I_3.$$

Thus, α, β, γ are the roots of the equation $I(\lambda)=0$.

Suppose all the roots are non-zero ($I_3 \neq 0$). In this case, as is known from Sec. 7-1, by transition to new coordinates the equation is reduced to the form

$$\alpha x^2+\beta y^2+\gamma z^2+\delta=0.$$

We find the coefficient δ using the invariant I_4. Namely:

$$\begin{vmatrix}\alpha & & & 0\\ & \beta & & \\ & & \gamma & \\ 0 & & & \delta\end{vmatrix}=\begin{vmatrix}a_{11} & \dots & a_{14}\\ \cdot & \cdot\ \cdot\ \cdot & \cdot\\ a_{41} & \dots & a_{44}\end{vmatrix}=I_4.$$

Whence

$$\delta=\frac{I_4}{\alpha\beta\gamma}=\frac{I_4}{I_3}.$$

Thus, if $I_3 \neq 0$ *then by transition to a new coordinate system, the equation is reduced to the form*

$$\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2+\frac{I_4}{I_3}=0,$$

where λ_1, λ_2, λ_3 are the roots of the equation $I(\lambda)=0$.

Let us now assume that one of the roots of the equation $I(\lambda)=0$ is equal to zero, the two others being different from zero. This will happen if $I_3=0$, but $I_2 \neq 0$. Then, by transition to new coordinates (see Sec. 7-1), the equation of the surface is reduced to one of the following forms

$$\alpha x^2+\beta y^2+2pz=0,$$
$$\alpha x^2+\beta y^2+\delta=0.$$

The first of them corresponds to the case $I_4 \neq 0$, and the second to the case $I_4=0$.

In the first case the coefficient p is found from the invariant I_4:

$$\begin{vmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & p \\ 0 & 0 & p & 0 \end{vmatrix} = -\alpha\beta p^2 = I_4,$$

and the equation of the surface will be

$$\lambda_1 x^2 + \lambda_2 y^2 + 2\sqrt{-\frac{I_4}{I_2}}\,z = 0.$$

In the case $I_4 = 0$ we change the coefficients of the equation by the quantities ε_{ij} so that $I_3 \neq 0$. Then, by transition to an appropriate system of coordinates, the equation is reduced to the form

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + \frac{I_4}{I_3} = 0.$$

Proceeding now to the limit as ε_{ij} tends to zero, we obtain the canonical form of the equation of our surface.

Example. Let $I_3 = I_4 = 0$, but

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0.$$

We put $\varepsilon_{33} = t$, and the remaining ε_{ij} equal to zero. Then

$$\frac{I_4(t)}{I_3(t)} = \frac{\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

The canonical form of the equation of the surface will be

$$\lambda_1 x^2 + \lambda_2 y^2 + \frac{\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = 0.$$

Finally, when two roots of the equation $I(\lambda) = 0$ are equal to zero, the equation of the surface is reduced to one of the forms:

$$ax^2 + 2pz = 0 \quad \text{or} \quad ax^2 + \delta = 0.$$

The coefficients p and δ are found by varying the coefficients of the equation of the surface as in the preceding case.

EXERCISES

1. Find the canonical form of the equation of a surface

$$(ax + by + cz + d)(a_1x + b_1y + c_1z + d_1) = 0.$$

2. Show that if $I_4 = 0$, then the surface represents either a cone, or a cylinder, or decomposes into a pair of planes.

3. Show that if $I_4 = 0$ and $I_3 = 0$, then the surface decomposes into a pair of planes.

Sec. 8-5. Diameters of a Curve, Diametral Planes of a Surface. The Centre of a Curve and a Surface

Let a quadric surface be specified by an equation in an arbitrary system of rectangular Cartesian coordinates

$$a_{11}x^2 + 2a_{12}xy + \ldots + a_{44} = 0. \qquad (*)$$

For the sake of brevity we introduce the following notation:

$$2F = a_{11}x^2 + 2a_{12}xy + \ldots + a_{44},$$
$$F_x = a_{11}x + a_{12}y + a_{13}z + a_{14},$$
$$F_y = a_{21}x + a_{22}y + a_{23}z + a_{24},$$
$$F_z = a_{31}x + a_{32}y + a_{33}z + a_{34}.$$

We already know (from Sec. 7-8) that the mid-points of the chords of a given direction $\lambda : \mu : \nu$, i.e. of those parallel to the line

$$\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu},$$

lie in the diametral plane. Form its equation if the surface is specified by the equation (*).

Let (x, y, z) be the mid-point of an arbitrary chord. The coordinates of the end-points may be written in the form

$$x_1 = x + \lambda t, \quad y_1 = y + \mu t, \quad z_1 = z + \nu t,$$
$$x_2 = x - \lambda t, \quad y_2 = y - \mu t, \quad z_2 = z - \nu t.$$

Substituting these coordinates into the equation of the surface, we get

$$2F(x, y, z) \pm 2t(\lambda F_x(x, y, z) + \mu F_y(x, y, z) + \nu F_z(x, y, z)) + t^2(a_{11}\lambda^2 + a_{12}\mu^2 + a_{33}\nu^2 + 2a_{12}\lambda\mu + 2a_{23}\mu\nu + 2a_{31}\nu\lambda) = 0.$$

It follows from this equation that the coefficient of t must be equal to zero:

$$\lambda F_x + \mu F_y + \nu F_z = 0. \qquad (**)$$

This is the equation of the diametral plane corresponding to the chords of the given direction $\lambda : \mu : \nu$.

If a surface has a centre, then each of the diametral planes passes through the centre. Consequently, the centre of a surface is determined from the equations

$$F_x = 0, \quad F_y = 0, \quad F_z = 0. \qquad (***)$$

Quadric curves are considered just in an analogous way. Here is the final result.

Suppose a curve is specified by the equation

$$2\Phi = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0.$$

We put

$$\Phi_x = a_{11}x + a_{12}y + a_{13},$$
$$\Phi_y = a_{21}x + a_{22}y + a_{23}.$$

Then *the diameter corresponding to the chords of the direction* $\lambda : \mu$, *i.e. to those parallel to the straight line*

$$\frac{x}{\lambda} = \frac{y}{\mu}.$$

is specified by the equation

$$\lambda\Phi_x + \mu\Phi_y = 0.$$

The centre of the curve (if any) is determined from the system of equations

$$\Phi_x = 0, \quad \Phi_y = 0.$$

EXERCISES

1. Show that if the origin is translated into the centre of the quadric curve

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0,$$

then its equation will take the form

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + \frac{I_3}{I_2} = 0.$$

2. Show that if the origin is translated into the centre of the quadric surface

$$a_{11}x^2 + 2a_{12}xy + \ldots + a_{44} = 0,$$

then the equation of the surface will take the form

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{31}zx + \frac{I_4}{I_3} = 0.$$

Sec. 8-6. Axes of Symmetry of a Curve. Planes of Symmetry of a Surface

Let us determine the planes of symmetry of a surface specified by an equation in arbitrary coordinates.

Suppose $\lambda : \mu : \nu$ is the direction perpendicular to the plane of symmetry. Since the mid-points of the chords of the direction $\lambda : \mu : \nu$ lie in the plane of symmetry, the latter is specified by the equation

$$\lambda F_x + \mu F_y + \nu F_z = 0. \qquad (*)$$

Since the direction $\lambda : \mu : \nu$ is perpendicular to the plane (*), then

$$\frac{a_{11}\lambda + a_{12}\mu + a_{13}\nu}{\lambda} = \frac{a_{21}\lambda + a_{22}\mu + a_{23}\nu}{\mu} = \frac{a_{31}\lambda + a_{32}\mu + a_{33}\nu}{\nu}. \qquad (**)$$

Finding $\lambda : \mu : \nu$ from this system of equations and substituting it into the equation (*), we get the equation of the plane of symmetry of the given surface.

To simplify the finding of $\lambda : \mu : \nu$ from the system (**), let us denote by ξ the common value of the three ratios (**). We then get an equivalent system

$$\left.\begin{aligned} (a_{11}-\xi)\lambda + a_{12}\mu + a_{13}\nu = 0, \\ a_{21}\lambda + (a_{22}-\xi)\mu + a_{23}\nu = 0, \\ a_{31}\lambda + a_{32}\mu + (a_{33}-\xi)\nu = 0. \end{aligned}\right\} \qquad (***)$$

Since λ, μ, ν are not equal to zero, we get

$$\begin{vmatrix} a_{11}-\xi & a_{12} & a_{13} \\ a_{21} & a_{22}-\xi & a_{23} \\ a_{31} & a_{32} & a_{33}-\xi \end{vmatrix} = I(\xi) = 0.$$

Whence we determine ξ and substitute it into the system (***) to find $\lambda : \mu : \nu$.

Knowing how to find the planes of symmetry of a surface, it is not difficult to find the coordinate system in which the equation of the surface has the canonical form.

Let us give an example.

Suppose that as a result of investigation of invariants of a surface the latter turned out to be an ellipsoid. Then its canonical equation will be

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + \frac{I_4}{I_3} = 0.$$

We see that the coordinate planes are the planes of symmetry of the surface.

If the roots ξ_1, ξ_2, ξ_3 of the equation $I(\xi) = 0$ are all different, then these planes are defined uniquely by the above mentioned method. But if there are equal roots among them, then this method yields no unique solution (the case of a surface of revolution), and to the requirement that the coordinate planes must be the planes of symmetry the condition of perpendicularity should be added.

Let us consider one more example. Suppose the surface is a hyperbolic paraboloid. In this case there are two and only two planes of symmetry. They are the coordinate axes. The origin is located at the point of intersection of the axis of the hyperboloid (the line of intersection of the planes of symmetry) with the surface.

A similar investigation of quadric curves results in the following:

The axes of symmetry of a quadric curve are specified by the equations

$$\lambda\Phi_x + \mu\Phi_y = 0$$

From the system

$$(a_{11} - \xi)\lambda + a_{12}\mu = 0,$$
$$a_{21}\lambda + (a_{22} - \xi)\mu = 0,$$

where ξ is the root of the equation $I(\xi) = 0$, we determine $\lambda : \mu$.

The system of coordinates in which the equation of the curve takes the canonical form is determined from the considerations analogous to those used for surfaces.

EXERCISES

1. Find the axis of the circular cone $x^2 + y^2 + z^2 - (ax + by + cz)^2 = 0$.
2. Find the vertex and the axis of the parabola $(ax + by + c)^2 + \alpha x + \beta y + \gamma = 0$.

Sec. 8-7. The Asymptotes of a Hyperbola. The Asymptotic Cone of a Hyperboloid

Suppose a hyperbola is specified by an equation in arbitrary coordinates

$$2\Phi = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0. \qquad (*)$$

Let us find the equation of its asymptotes, for which purpose we pass over to a new system of coordinates $x'y'$ in which the equation of the hyperbola has the canonical form:

$$2\Phi' = \alpha x'^2 + \beta y'^2 + \gamma = 0.$$

As we know (from sec 3-4), in this system of coordinates both asymptotes are specified by the equation

$$\alpha x'^2 + \beta y'^2 = 0, \quad \text{i.e.} \quad 2\Phi' - \gamma = 0.$$

If we now come back to the coordinates xy, then for the hyperbola we shall obtain once again the equation (*), and consequently the equation $2\Phi - \gamma = 0$ for its asymptotes.

The constant γ, as is known from Sec. 8-3, is equal to I_3/I_2. Hence, *the equation of the asymptotes of a hyperbola specified by an equation in the general form will be*

$$2\Phi - \frac{I_3}{I_2} = 0.$$

Reasoning just in the same way when considering a hyperboloid (of one or two sheets)

$$2F = a_{11}x^2 + 2a_{12}xy + \ldots + a_{44} = 0,$$

we find the equation of its asymptotic cone

$$2F - \frac{I_4}{I_3} = 0.$$

EXERCISES

1. Find the asymptotes of the hyperbola $(ax + by + c)(a_1x + b_1y + c_1) = \text{const}$.
2. Find the asymptotes of the hyperbola $\lambda(ax + by + c)^2 + \mu(a_1x + b_1y + c_1)^2 = \nu$, $\lambda\mu < 0$.

Sec. 8-8. A Tangent Line to a Curve. A Tangent Plane to a Surface

Let a quadric curve be specified by an equation of the general form

$$2\Phi = a_{11}x^2 + 2a_{12}xy + \ldots + a_{33} = 0.$$

Let us form the equation of its tangent line at an arbitrary point $A_0(x_0, y_0)$.

A *tangent line to a curve* is defined as the limit of a secant g when the point K infinitely approaches A_0 (Fig. 88).

Let A (x, y) be an arbitrary point on the tangent line. We denote by A' (x', y') the point of the secant nearest to A. Obviously, when $K \to A_0$, $A' \to A$.

The coordinates of the point K in terms of the coordinates of A_0 and A' may be written in the form

$$x_K = x_0 + t\,(x' - x_0),$$
$$y_K = y_0 + t\,(y' - y_0).$$

Substituting the coordinates of the point K into the equation of the curve, we get

$$2\Phi\,|_K = 2\,\Phi|_{A_0} + 2t\,\{(x' - x_0)\,\Phi_x\,|_{A_0} + (y' - y_0)\,\Phi_y\,|_{A_0}\} + t^2\,\{a_{11}\,(x' - x_0)^2 + 2a_{12}\,(x' - x_0)\,(y' - y_0) + a_{22}\,(y' - y_0)^2\} = 0,$$

where the subscript A_0 indicates that the coordinates of the point A_0 should be taken as x and y. Since the point

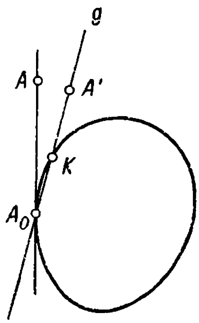


Fig. 88.

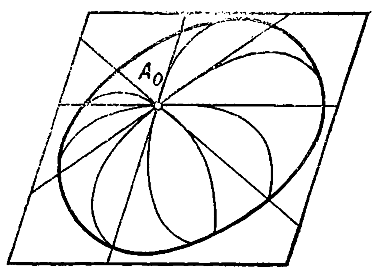


Fig. 89.

A_0 lies on the curve, $\Phi\,|_{A_0} = 0$. Therefore the equation may be reduced by t. We obtain

$$2\,(x' - x_0)\,\Phi_x\,(x_0, y_0) + 2\,(y' - y_0)\,\Phi_y\,(x_0, y_0) + t\,\{a_{11}\,(x' - x_0)^2 + 2a_{12}\,(x' - x_0)\,(y' - y_0) + a_{22}\,(y' - y_0)^2\} = 0.$$

Let now $K \to A_0$. Then $t \to 0$, and $A' \to A$ (i.e. $x' \to x$, $y' \to y$), and we get

$$(x - x_0)\,\Phi_x\,(x_0, y_0) + (y - y_0)\,\Phi_y\,(x_0, y_0) = 0. \qquad (*)$$

This equation is linear with respect to x and y, and therefore this is an equation of a straight line. An arbitrary point A of the tangent line satisfies it. Hence, this is the equation of the tangent line.

A *tangent plane to a surface* at point A_0 is defined as the plane containing the tangent lines to all the curves on the surface emanating from A_0 (Fig. 89). The equation of the tangent plane to a quadric surface at point $A_0\,(x_0, y_0, z_0)$ will be:

$$2F = a_{11}x^2 + 2a_{12}xy + \\ + \ldots + a_{44} = 0.$$

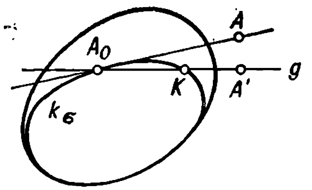


Fig. 90.

Draw an arbitrary plane σ through the point A_0. It will cut the surface in a quadric curve k_σ. Draw a line tangent to the curve k_σ at point A_0 and denote by $A\,(x, y, z)$ an arbitrary point on this tangent (Fig. 90).

Take a point K on k_σ close to A_0, and draw a secant g through the points A and K. Let $A'\,(x', y', z')$ be the point on the secant nearest to A. Obviously, if $K \to A_0$, then $A' \to A$.

The coordinates of the point K in terms of coordinates of A_0 and A' may be expressed in the form $x_K = x_0 + t\,(x' - x_0)$, $y_K = y_0 + t\,(y' - y_0)$, $z_K = z_0 + t\,(z' - z_0)$.

Substituting the coordinates of K into the equation of the surface, we get

$$2F\,|_{A_0} + 2t\,\{(x' - x_0)\,F_x\,|_{A_0} + (y' - y_0)\,F_y\,|_{A_0} + \\ + (z' - z_0)\,F_z\,|_{A_0}\} + t^2\,\{a_{11}\,(x' - x_0)^2 + \\ + 2a_{12}\,(x' - x_0)\,(y' - y_0) + \ldots\} = 0. \qquad (**)$$

But $2F\,|_{A_0} = 0$, since the point A_0 is situated on the surface. Dividing the equation (**) by t and proceeding to the limit as K tends to A_0, we obtain

$$(x - x_0)\,F_x\,|_{A_0} + (y - y_0)\,F_y\,|_{A_0} + (z - z_0)\,F_z\,|_{A_0} = 0.$$

This equation is linear with respect to x, y, z and therefore specifies a plane. Since it is satisfied by the coordinates of any point A on the tangent k_σ at point A_0 whatever σ is, it represents the equation of a tangent plane to a surface at point A_0.

EXERCISES

1. Show that a tangent plane to a quadric surface at point P is parallel to the diametral plane corresponding to the chords parallel to the diameter passing through P.

2. Let $2\Phi = a_{11}x^2 + 2a_{12}xy + \ldots + a_{33} = 0$ be a quadric curve, and $A_0(x_0, y_0)$ a point outside this curve. Draw an arbitrary line g through A_0. Let $A(x, y)$ be an arbitrary point on this line. The coordinates of any point B of the line g may be represented in the form

$$x_B = x_0 + t(x - x_0), \quad y_B = y_0 + t(y - y_0).$$

The values of the parameter t corresponding to the points B_1 and B_2 of intersection of the curve $2\Phi = 0$ with the line g are found from the quadric equation

$$2\Phi(x_0 + t(x - x_0), \quad y_0 + t(y - y_0)) = 0. \qquad (***)$$

When the line g approaches the tangent, the roots of the equation (***) merge. Taking into consideration this fact, form the equation of the pair of tangent lines to a quadric curve emanating from the point A_0.

3. Form the equation of a cone with the vertex $A_0(x_0, y_0, z_0)$ touching the quadric surface $2F = 0$.

4. Form the equation of a cylinder with the axis parallel to the straight line

$$x/\lambda = y/\mu = z/\nu.$$

The cylinder is circumscribed about the quadric surface $2F = 0$.

5. Show that the tangent plane to a hyperboloid of one sheet and to a hyperbolic paraboloid intersects the surface along two straight lines.

6. Show that the confocal quadric surfaces

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1,$$

passing through point (x_0, y_0, z_0) intersect at this point at right angles. It is assumed that the point does not lie in any of the coordinate planes.

Chapter 9

Linear Transformations

Sec. 9-1. Orthogonal Transformations

Suppose an arbitrary figure F is carried into a figure F' by motion, or by motion and mirror reflection. Then the figure F' is said to be obtained from F by an orthogonal transformation. Obviously, the *orthogonal transformation of a figure leaves the distances between its points unchanged.*

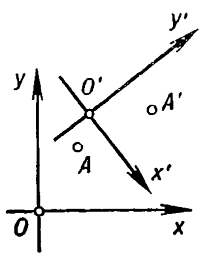


Fig. 91.

Let us find the formulas which establish the relationship between the coordinates of an arbitrary point A (x, y, z) of the figure F and the corresponding point A' (x', y', z') of the figure F'.

Let us imagine that the coordinate system s (x, y, z) is rigidly connected with the figure F. Then, as a result of an orthogonal transformation, it will go into a system of coordinates s' with reference to which the coordinates of the point A' will be x, y, z (Fig. 91). Thus, the problem consists in that we have to express the coordinates of the point A' in the coordinate system s if its coordinates in the system s' are known.

As is known (from Sec. 5-4), the relationship between the coordinates of a point with reference to two systems of rectangular Cartesian coordinates is established by the formulas

$$\left.\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z + a_{14}, \\ y' &= a_{21}x + a_{22}y + a_{23}z + a_{24}, \\ z' &= a_{31}x + a_{32}y + a_{33}z + a_{34}, \end{aligned}\right\} \qquad (*)$$

whose coefficients satisfy the following conditions

$$\left.\begin{array}{ll} a_{11}^2+a_{21}^2+a_{31}^2=1, & a_{11}a_{12}+a_{21}a_{22}+a_{31}a_{32}=0 \\ a_{12}^2+a_{22}^2+a_{32}^2=1, & a_{12}a_{13}+a_{22}a_{23}+a_{32}a_{33}=0, \\ a_{13}^2+a_{23}^2+a_{33}^2=1, & a_{13}a_{11}+a_{23}a_{21}+a_{33}a_{31}=0. \end{array}\right\} \quad (**)$$

Hence, taking into consideration all this, we come to a conclusion that *any orthogonal transformation is specified by the formulas* (*) *whose coefficients satisfy the conditions* (**).

Let us show the converse that *any transformation specified by the formulas* (*) *under the conditions* (**) *is an orthogonal transformation*, i.e. the transformed figure is obtained from the given one by motion, or by motion and mirror reflection.

Let A_1 (x_1, y_1, z_1) and A_2 (x_2, y_2, z_2) be two arbitrary points of the figure F, and A_1' (x_1', y_1', z_1') and A_2' (x_2', y_2', z_2') the corresponding points of the figure F'. The square of the distance between the points A_1' and A_2' is equal to

$$(x_1'-x_2')^2+(y_1'-y_2')^2+(z_1'-z_2')^2.$$

Substituting the expressions for x_1', x_2', y_1', y_2', z_1', z_2' according to the formulas (*), and taking advantage of the conditions (**), we get

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2.$$

Hence, the distance between any two points of the figure F is equal to the distance between the corresponding points of the figure F'. Consequently, F is congruent to F', and F' is obtained from F by motion, or by motion and mirror reflection.

Orthogonal transformations possess the following geometrically obvious properties which, however, may be verified with the aid of the formulas (*):

1. *The successive application of two orthogonal transformations is an orthogonal transformation once again*, i.e. if figure F' is obtained by an orthogonal transformation from F, and figure F'' by an orthogonal transformation from F', then F'' is obtained by an orthogonal transformation from F.

2. *The inverse of an orthogonal transformation is itself an orthogonal transformation*, i.e. if F' is obtained by an orthogonal transformation from F, then F is obtained by an orthogonal transformation from F'.

3. *An identity transformation, i.e. the transformation specified by the formulas*

$$x' = x, \quad y' = y, \quad z' = z,$$

is an orthogonal transformation.

Orthogonal transformations in the plane are defined similarly. They possess analogous properties and are specified by the formulas

$$\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}, \\ y' &= a_{21}x + a_{22}y + a_{23}, \end{aligned}$$

whose coefficients satisfy the following conditions

$$\begin{aligned} a_{11}^2 + a_{21}^2 &= 1, \\ a_{12}^2 + a_{22}^2 &= 1, \end{aligned} \quad a_{11}a_{12} + a_{21}a_{22} = 0.$$

Since the formulas for transforming rectangular Cartesian coordinates (see Sec. 2-7) coincide with the formulas of orthogonal transformations, then from the results of Sec. 3-8 concerning the reduction of the equations of quadric curves to the canonical form, it follows that any quadric curve can be transformed by an orthogonal transformation into a curve of one of the following types

$$\begin{aligned} \alpha x^2 + \beta y^2 + \gamma &= 0, \\ \alpha x^2 + \beta y^2 &= 0, \\ \alpha x^2 + 2py &= 0, \\ \alpha x^2 + q &= 0, \\ x^2 &= 0. \end{aligned}$$

EXERCISES

1. Form the formulas of the orthogonal transformation which carries the plane xy (yz, xz) into itself, and the plane xy into the plane xz (yz).

2. Form the formulas of the orthogonal transformation which leaves the origin in its place and brings the x-axis into the straight line

$$\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}.$$

Sec. 9-2. Affine Transformations

Orthogonal transformations are a particular case of more general transformations of figures, the so-called *affine transformations.* The latter are specified by the formulas

$$\left.\begin{aligned} x' &= a_{11}x+a_{12}y+a_{13}z+a_{14}, \\ y' &= a_{21}x+a_{22}y+a_{23}z+a_{24}, \\ z' &= a_{31}x+a_{32}y+a_{33}z+a_{34}, \end{aligned}\right\} \qquad (*)$$

where the coefficients a_{ij} are any real numbers satisfying the only condition

$$\Delta=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\neq 0. \qquad (**)$$

Obviously, *this definition is invariant with respect to the coordinate system chosen,* since the coordinates of a point in one coordinate system are expressed linearly in terms of its coordinates in any other system of coordinates.

Affine transformations possess the following properties which are easily checked:

1. *The successive application of two affine transformations is an affine transformation.*

2. *The inverse of an affine transformation is also an affine transformation.*

3. *The identity transformation is affine.*

All these properties are easily verified with the aid of the formulas $(*)$. Let us, for instance, check the second property.

Solving the system of equations (*) with respect to x, y, z (the determinant of the system is non-zero), we get

$$\left.\begin{aligned} x &= a'_{11}x' + a'_{12}y' + a'_{13}z' + a'_{14}, \\ y &= a'_{21}x' + a'_{22}y' + a'_{23}z' + a'_{24}, \\ z &= a'_{31}x' + a'_{32}y' + a'_{33}z' + a'_{34}, \end{aligned}\right\} \qquad (***)$$

where a'_{ij} (for i, $j \leqslant 3$) represent the reduced algebraic cofactors of the elements a_{ij} in Δ. As is known, the determinant Δ' formed from a'_{ij} is equal to $\Delta^{-1} \neq 0$. Whence it follows that the transformation associating the point (x, y, z) with the point (x', y', z') according to the formulas (***), i.e. the transformation inverse to the affine one (*) is affine.

We conclude with an important note that *an affine transformation is defined uniquely if there given the images of four points not lying in one plane.* Indeed, substituting the coordinates of the given four points and their images into the first of the equations (*), we get

$$\begin{aligned} x'_1 &= a_{11}x_1 + a_{12}y_1 + a_{13}z_1 + a_{14}, \\ x'_2 &= a_{11}x_2 + a_{12}y_2 + a_{13}z_2 + a_{14}, \\ x'_3 &= a_{11}x_3 + a_{12}y_3 + a_{13}z_3 + a_{14}, \\ x'_4 &= a_{11}x_4 + a_{12}y_4 + a_{13}z_4 + a_{14}. \end{aligned}$$

These equalities may be considered as a system of equations with respect to a_{11}, a_{12}, a_{13}, a_{14}. The determinant of the system

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = -\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix}$$

is equal (by absolute value) to six times the volume of a tetrahedron with the vertices at the given four points and, consequently, is non-zero. Hence, the quantities a_{11}, a_{12}, a_{13}, a_{14} are defined uniquely from this system. It is proved in a similar way that the coefficients of two other formulas (*) are also defined uniquely.

An affine transformation in the plane is defined uniquely if there are given the images of three points not lying on a straight line.

EXERCISES

1. Derive the formulas of the affine transformation which carries the points (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1) into the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4).
2. Derive the formulas of the affine transformation in the plane transferring the coordinate axes x and y into the two given lines

$$ax + by + c = 0, \quad a_1x + b_1y + c_1 = 0.$$

Sec. 9-3. The Affine Transformation of a Straight Line and a Plane

From the single-valued solvability of the formulas of the affine transformation

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14},$$
$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24},$$
$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34},$$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0. \qquad (*)$$

with respect to x, y, and z it follows that *the affine transformation carries different points into different points, and that any point (x', y', z') is an image of some point (x, y, z).*

Let us prove that *the affine transformation carries a plane into a plane, a straight line into a straight line, preserving parallelism.*

Suppose σ is an arbitrary plane and

$$ax + by + cz + d = 0 \qquad (**)$$

is its equation. Under the affine transformation (*) the plane σ goes into some figure σ'. Since the coordinates of each point of σ satisfy the equation (**) and are linearly expressed in terms of the coordinates of the corresponding point of the figure σ', then the coordinates of the points belonging to σ' also satisfy the linear equation

$$a'x' + b'y' + c'z' + d' = 0, \qquad (**')$$

which is obtained from (**) by replacing x, y, z by their linear expressions with respect to x', y', z' according to the formulas (***) of Sec. 9-2. The equation (**') cannot be an identity, since introducing in it the variables x, y, z (instead of x', y', z') according to the formulas (*), we must get (**) once again.

Hence, σ' lies in the plane specified by the equation (**'). Let us show that σ' coincides with this plane. Indeed, let (x', y', z') be any point belonging to the plane (**'). Under the affine transformation inverse to (*) its image satisfies (**) and, consequently, belongs to σ'. Whence we conclude that σ' coincides with the plane (**') (and is not its portion). This proves that under the affine transformation a plane goes into a plane.

Since under the affine transformation a plane goes into a plane and the inverse of the affine transformation is an affine transformation, *different planes go into different planes.*

Since under the affine transformation different points are carried into different points, then *parallel planes are carried into parallel planes.*

Since through a straight line there can be drawn two different planes, and under the affine transformation different planes go into different planes, then *under the affine transformation a straight line goes into a straight line.*

Since two parallel lines can be defined by the intersection of two parallel planes with a third plane, and parallel planes under the affine transformation go into parallel planes, then *the affine transformation carries parallel lines into parallel lines.*

Let us note in conclusion that the affine transformations in the plane possess analogous properties. In particular, *under the affine transformation in the plane straight lines go into straight lines, and parallelism is preserved.*

EXERCISES

1. Find the planes into which the coordinate planes xy, yz, zx will go under the affine transformation (*).
2. Find the straight lines into which the coordinate axes will be carried under the affine transformation (*).

Sec. 9-4. The Principal Invariant of the Affine Transformation

Under the orthogonal transformation the distance between points remains unchanged. In this connection the distance between the points is an invariant of the orthogonal transformation. We could mention many other invariants of the orthogonal transformation, for instance, the angle between straight lines, or the area of a triangle. The distance between points is not only the simplest but also the principal invariant, since the rest of the invariants can be expressed in its terms.

Under the affine transformation the distance between points, as a rule, undergoes a change, therefore the distance between points is not an invariant of the general affine transformation.

An affine ratio of three points on a straight line is the simplest and principal invariant of the affine transformation. The affine ratio of the points A, B, C on a straight line is defined as the number

$$(ABC) = \frac{AB}{BC}.$$

Let us show that the affine ratio of three points on a straight line remains unchanged under the affine transformation, i.e. if under an affine transformation the points A, B, C go into the points A', B', C', then

$$(ABC) = (A'B'C').$$

In general, we may consider that the points A, B, C lie on the x-axis (the straight line AB may be taken for the x-axis). Furthermore, we may consider that the points A', B', C' also lie on the x-axis, since applying an orthogonal transformation which obviously leaves the affine ratio unchanged (preserving the lengths of line segments) the three points A', B', C' can always be transferred onto the x-axis. In this case we have

$$(ABC)=\frac{|x_A-x_B|}{|x_B-x_C|}, \quad (A'B'C')=\frac{|x_{A'}-x_{B'}|}{|x_{B'}-x_{C'}|}.$$

But the coordinates x' of the points A', B', C' are related with the coordinates x of the points A, B, C by the equation

$$x' = a_{11}x + a_{14},$$

and the equality of the affine ratios (ABC) and $(A'B'C')$ are checked in an obvious way.

EXERCISES

1. Show that there exists an affine transformation which maps a given arbitrary triangle into a regular one. Show that the point of intersection of medians goes into the point of intersection of medians.
2. Show that under an affine transformation any given parallelogram can be mapped into a square. Is it possible to map any quadrilateral into a square applying an affine transformation?
3. Under what condition does the affine transformation of a plane specified by the formulas (*) of the preceding section leave some point fixed?

Sec. 9-5. Affine Transformations of Quadric Curves and Surfaces

Since the quadric curve is defined as a locus of points whose Cartesian coordinates satisfy an equation of the second degree, and the coordinates of a point are linearly expressed in terms of the coordinates of its image under the affine transformation, then *under the affine transformation a quadric curve goes into a quadric curve.*

Analogously, *under the affine transformation a quadric surface is mapped onto a quadric surface.*

Since the affine transformation carries parallel lines into parallel lines and preserves the affine ratio of three points, in particular, the mid-point of a line segment goes into the mid-point of a line segment, then *under the affine transformation the diameters of a quadric curve are carried into the diameters, moreover, conjugate diameters go into conjugate diameters, the centre being mapped into centre.*

Under the affine transformation quadric surfaces possess analogous properties.

Since under the affine transformation real points go into real points and imaginary points into imaginary points, the *affine transformation carries a real curve into a real curve, and an imaginary curve into an imaginary curve.*

Obviously, if a figure is finite, then under the affine transformation its image is a finite figure; if a figure is infinite, then its image is also an infinite figure.

Taking into account the above mentioned properties of the affine transformation, we may conclude:

Under any affine transformation an ellipse is mapped into an ellipse, a hyperbola into hyperbola, a parabola into parabola, a pair of intersecting lines into a pair of intersecting lines, and a pair of parallel lines into a pair of parallel lines.

Analogous conclusions may be formulated for quadric surfaces.

Two figures are said to be *affinely equivalent* if under an affine transformation they can be mapped into each other.

All ellipses are affinely equivalent to a circle

$$x^2 + y^2 = 1.$$

All hyperbolas are affinely equivalent to an equilateral hyperbola

$$x^2 - y^2 = 1.$$

All parabolas are affinely equivalent to the parabola

$$y = x^2.$$

Let us prove, for instance, the first assertion. Under the orthogonal transformation any ellipse can be carried into the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

And this ellipse, by uniform compression (elongation) relative to the coordinate axes

$$x'=\frac{x}{a}, \quad y'=\frac{y}{b},$$

is transformed into a circle

$$x'^2+y'^2=1.$$

When considering space, we may formulate similar assertions concerning the affine equivalence of quadric surfaces.

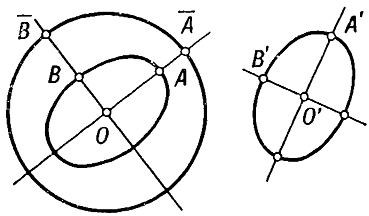


Fig. 92.

Finally, we are going to show that *any affine transformation on the plane can be obtained by successive application of three transformations: a uniform elongation (compression) relative to two mutually perpendicular lines, and an orthogonal transformation.*

The proof is rather simple. Under the affine transformation the circle

$$x^2+y^2=1$$

will go into an ellipse E' (Fig. 92). Let A' and B' be its two successive vertices, O' its centre, $\bar{A}$ and $\bar{B}$ the corresponding points of the circle. The straight lines $O\bar{A}$ and $O\bar{B}$ are mutually perpendicular, since they are conjugate diameters of the circle (in fact, they correspond to the conjugate diameters $O'A'$, $O'B'$ of the ellipse).

We now introduce two systems of coordinates: $\bar{x}\bar{y}$, taking the lines $O\bar{A}$ and $O\bar{B}$ for positive semi-axes $\bar{x}$ and $\bar{y}$; and $x'y'$, taking the lines $O'A'$, $O'B'$ for the positive semi-axes. In the coordinate system $x'y'$ the ellipse E' is specified by the equation

$$\alpha x'^2 + \beta y'^2 = 1.$$

There exists an orthogonal transformation which carries the ellipse E

$$\alpha\bar{x}^2 + \beta\bar{y}^2 = 1$$

into the ellipse E', its vertices A and B being transformed into the vertices A' and B' of the ellipse E'.

Let us now consider the affine transformation which consists of a uniform elongation (compression) with respect to the $\bar{y}$-axis under which the point $\bar{A}$ goes into A, a uniform elongation (compression) relative to the $\bar{x}$-axis under which the point $\bar{B}$ goes into B, and an orthogonal transformation under which the ellipse E is carried into E'. The affine transformation designed in such a way carries the points O, $\bar{A}$, $\bar{B}$ into the points O, A', B', like the given one, and, consequently, coincides with it (see Sec. 9-2). The assertion has been proved.

An analogous assertion may be formulated for an affine transformation in space. Namely, *any affine transformation in space can be decomposed into three uniform compressions (elongations) with respect to three mutually perpendicular directions, and an orthogonal transformation.*

EXERCISES

1. Derive the properties of the conjugate diameters of an ellipse from the properties of diameters of a circle. Derive the properties of the diameters and diametral planes of an ellipsoid from the properties of the diameters and diametral planes of a sphere.

2. An affine transformation in the plane is specified by the formulas

$$\begin{aligned} x' &= a_1x + b_1y + c_1, \\ y' &= a_2x + b_2y + c_2. \end{aligned}$$

As is shown, this transformation can be decomposed into a uniform elongation (compression) with respect to two mutually perpendicular directions and an orthogonal transformation. Find the coefficients of elongation (compression).

Sec. 9-6. Projective Transformations

Affine transformations of figures are a particular case of more general, so-called *projective transformations* specified by the formulas

$$\left.\begin{aligned} x' &= \frac{a_{11}x + a_{12}y + a_{13}z + a_{14}}{a_{41}x + a_{42}y + a_{43}z + a_{44}}, \\ y' &= \frac{a_{21}x + a_{22}y + a_{23}z + a_{24}}{a_{41}x + a_{42}y + a_{43}z + a_{44}}, \\ z' &= \frac{a_{31}x + a_{32}y + a_{33}z + a_{34}}{a_{41}x + a_{42}y + a_{43}z + a_{44}}, \end{aligned}\right\} \qquad (*)$$

whose coefficients satisfy the only condition:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \neq 0.$$

These formulas define the transformation for any figure F which does not intersect the plane σ_∞:

$$a_{41}x + a_{42}y + a_{43}z + a_{44} = 0.$$

In our further considerations we shall assume that the figure under transformation does not intersect with the plane σ_∞.

Obviously, *this definition of the projective transformation is invariant with respect to the choice of the coordinate system.*

A direct check may convince us that *the successive application of two projective transformations is a projective transformation, the inverse of a projective transformation is again a projective transformation, the identity transformation is also a projective transformation.*

The projective transformation possesses many properties of the affine transformation. In particular, under projective transformations points lying on a straight line are mapped into points lying on a straight line.

The affine ratio of three points, generally speaking, is not preserved under the projective transformation, but in return, the *anharmonic ratio of four points on a straight line is preserved.* This ratio is defined in the following way.

Suppose A, B, C, D are four points on a straight line, and $\boldsymbol{e}$ is a non-zero vector which is not perpendicular to the line. Then the *anharmonic* ratio of the points A, B, C, D (taken in the given succession) is defined as the number

$$(ABCD)=\frac{\boldsymbol{e}\cdot\overrightarrow{AC}}{\boldsymbol{e}\cdot\overrightarrow{BC}}:\frac{\boldsymbol{e}\cdot\overrightarrow{AD}}{\boldsymbol{e}\cdot\overrightarrow{BD}}.$$

Obviously, this definition is invariant with reference to the choice of the vector $\boldsymbol{e}$. Therefore, taking the basis vector $\boldsymbol{e}_x$ as $\boldsymbol{e}$, we obtain

$$(ABCD)=\frac{x_C-x_A}{x_C-x_B}:\frac{x_D-x_A}{x_D-x_B}. \qquad (**)$$

provided the x-axis is not perpendicular to the line AD.

If the y- and z-axes are not perpendicular to the straight line, then analogous formulas are obtained with the coordinates y and z.

Let us show that the *anharmonic ratio of the four points A, B, C, D on a straight line is preserved under the projective transformation.*

Generally speaking, we may consider that the points A, B, C, D lie on the x-axis (the line AD may be taken for the x-axis). Furthermore, we may consider that their images A', B', C', D' also lie on the x-axis, since under the orthogonal transformation which, obviously, does not change the anharmonic ratio, they can be mapped onto the x-axis. And the coordinates x' of the points A', B', C', D' are expressed in terms of the coordinates x of the points A, B, C, D by the formula

$$x'=\frac{a_{11}x+a_{14}}{a_{41}x+a_{44}},$$

and through a direct check we get sure that

$$\frac{x_C - x_A}{x_C - x_B} : \frac{x_D - x_A}{x_D - x_B} = \frac{x_{C'} - x_{A'}}{x_{C'} - x_{B'}} : \frac{x_{D'} - x_{A'}}{x_{D'} - x_{B'}},$$

i.e.

$$(ABCD) = (A'B'C'D'),$$

which was required to be proved.

Projective transformations in the plane are specified by the following formulas

$$x' = \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}}, \quad y' = \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0 \qquad (***)$$

and possess similar properties.

The term "projective transformation" is linked with the following property of these transformations.

Any figure F' contained in a plane α obtained from figure F of the same plane by means of a projective transformation not reduced to the affine transformation can be obtained by central projecting from a centre S of figure $\overline{F}$ congruent to F.

Conversely, *any figure obtained by central projecting can be obtained from F under a projective transformation.*

We are going to prove only the second part of the assertion. Without loss of generality we may consider that the xy-plane is the plane α.

Let A $(x, y, 0)$ be an arbitrary point of the figure F, $\overline{A}$ $(\overline{x}, \overline{y}, \overline{z})$ the corresponding point of $\overline{F}$, S (x_0, y_0, z_0) the centre of projecting, and A' $(x', y', 0)$ the projection of $\overline{A}$ from the centre S on the xy-plane. Since the points S, $\overline{A}$, and A' are collinear, we get

$$\frac{x' - x_0}{\overline{x} - x_0} = \frac{y' - y_0}{\overline{y} - y_0} = \frac{-z_0}{\overline{z} - z_0}.$$

Whence

$$x' = \frac{-z_0\overline{x} + \overline{z}x_0}{\overline{z} - z_0}, \quad y' = \frac{-z_0\overline{y} + \overline{z}y_0}{\overline{z} - z_0}.$$

Since $\bar{x}$, $\bar{y}$, and $\bar{z}$ are linearly expressed in terms of x and y (figure $\bar{F}$ is obtained from F via an orthogonal transformation), the expressions for x' and y' in terms of x and y will have the form (***) which means that the figure F' obtained by projecting can also be obtained by means of a projective transformation of the figure F.

EXERCISES

1. Show that a projective transformation in the plane is defined uniquely if it is specified for four points no three of which are collinear.
2. Express in terms of the anharmonic ratio $(ABCD)$ the anharmonic ratios of these points taken in any other order, for instance, $(ABCD)$, $(BACD)$, and so on.

Sec. 9-7. Homogeneous Coordinates. Supplementing a Plane and a Space with Elements at Infinity.

In a plane, the *homogeneous coordinates* of a point, whose Cartesian coordinates are x and y, are any three numbers x_1, x_2, x_3 (not all equal to zero) for which

$$x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3}.$$

The homogeneous coordinates of a point are defined not uniquely. Namely, if x_1, x_2, x_3 are the homogeneous coordinates of a point, then ρx_1, ρx_2, ρx_3 $(\rho \neq 0)$ will also be the homogeneous coordinates of this point.

Since in Cartesian coordinates any straight line is specified by the equation

$$a_1 x + a_2 y + a_3 = 0 \quad (a_1^2 + a_2^2 \neq 0)$$

and any equation of this form is the equation of a straight line, then any straight line is specified in homogeneous coordinates by the equation

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0 \quad (a_1^2 + a_2^2 \neq 0)$$

and any such equation is the equation of a straight line.

For any point (x, y) in a plane it is obviously possible to find three numbers which will be its homogeneous coordinates, for instance, x, y, 1. The converse is, generally speaking, false. Namely, for the three points x_1, x_2, x_3, where $x_3 = 0$, it is impossible to indicate a point for which these numbers would be its homogeneous coordinates. This circumstance causes many inconveniences when considering a number of problems concerning, in particular, projective transformations of figures. In connection with this, we shall supplement the plane with new elements, namely, *infinite points and a straight line at infinity.*

Hence, we shall consider that to the three numbers x_1, x_2, x_3 there corresponds a point at infinity if $x_3 = 0$. The locus of infinite points will be called a *straight line at infinity.*

In a plane extended in such a way any equation of the form

$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$

is the equation of a straight line. If $a_1 = a_2 = 0$, *then the line is at infinity.*

In an extended plane any two straight lines intersect, since the system of two linear equations

$$\left.\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 = 0, \\ b_1x_1 + b_2x_2 + b_3x_3 = 0 \end{aligned}\right\} \qquad (*)$$

always has a non-trivial solution (x_1, x_2, x_3 are not all equal to zero). In particular, *two parallel lines intersect in a point at infinity.* Indeed, if the straight lines (*) are parallel, then

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \lambda.$$

Multiplying the second equation of the system (*) by λ and subtracting it from the first equation, we get $(a_3 - \lambda b_3)\, x_3 = 0$, whence $x_3 = 0$.

The projective transformation of figures introduced in the preceding section can be continued on an extended plane. Namely, let us consider on an extended plane a transformation specified by the formulas

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \end{aligned} \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0.$$

On a non-extended plane this transformation coincides with the projective transformation introduced before. Indeed, on a non-extended plane

$$x_3 \neq 0, \quad x_3' \neq 0.$$

Dividing the first two formulas by the third one termwise, we get

$$x' = \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}},$$
$$y' = \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}}.$$

For a space the homogeneous coordinates x_1, x_2, x_3, x_4 of a point are introduced analogously as the four numbers related with the Cartesian coordinates as follows

$$x = \frac{x_1}{x_4}, \qquad y = \frac{x_2}{x_4}, \qquad z = \frac{x_3}{x_4}.$$

In just the same way a space is also supplemented with elements at infinity: *infinite points*, *infinite straight lines*, and *an infinite plane*. And it turns out that *in a space supplemented with elements at infinity any equation*

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$$

specifies a plane (*an infinite one* if $a_1 = a_2 = a_3 = 0$), *any two independent equations*

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0,$$
$$b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 = 0$$

define a straight line $\left(\textit{possibly at infinity if } \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}\right)$.

The projective transformations defined in the preceding section are spread to an extended space and are specified in homogeneous coordinates by the formulas

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4, \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4, \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4, \\ x_4' &= a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4, \end{aligned}$$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \neq 0.$$

EXERCISES

1. Derive the formulas for the projective transformation of an extended plane which carries the straight lines $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ into the straight lines

$$\begin{aligned} a_1x_1 + b_1x_2 + c_1x_3 &= 0, \\ a_2x_1 + b_2x_2 + c_2x_3 &= 0, \\ a_3x_1 + b_3x_2 + c_3x_3 &= 0. \end{aligned}$$

2. Find the coordinates of the point of intersection of the straight lines

$$\frac{x_1\alpha_4 - x_4\alpha_1}{k_1} = \frac{x_2\alpha_4 - x_4\alpha_2}{k_2} = \frac{x_3\alpha_4 - x_4\alpha_3}{k_3},$$

$$\frac{x_1\beta_4 - x_4\beta_1}{k_1} = \frac{x_2\beta_4 - x_4\beta_2}{k_2} = \frac{x_3\beta_4 - x_4\beta_3}{k_3}.$$

Sec. 9-8. The Projective Transformations of Quadric Curves and Surfaces

In homogeneous coordinates a quadric curve is, obviously, specified by the equation

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + \dots + a_{33}x_3^2 = 0, \qquad (*)$$

which is obtained from its equation in Cartesian coordinates

$$a_{11}x^2 + 2a_{12}xy + \ldots + a_{33} = 0 \qquad (**)$$

by replacing x by $\frac{x_1}{x_3}$ and y by $\frac{x_2}{x_3}$.

Let us supplement a plane with infinite elements and continue the curve specified by the equation (*) on the extended plane by joining to it all ideal points (if any) which satisfy the equation (*).

Let us show that *on an extended plane a quadric curve is projectively equivalent to one of the following simple curves*:

$$\left.\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= 0,\\ x_1^2 - x_2^2 + x_3^2 &= 0,\\ x_1^2 + x_2^2 &= 0,\\ x_1^2 - x_2^2 &= 0,\\ x_1^2 &= 0, \end{aligned}\right\} \qquad (***)$$

i.e. can be mapped by a projective transformation into one of them.

Considering the reduction of a quadric curve to the canonical form (Sec. 3-8), we showed that there exists a system of coordinates $x'y'$ in which the equation of the curve (**) takes one of the following forms:

$$\begin{aligned} \alpha x'^2 + \beta y'^2 + \gamma &= 0,\\ \alpha x'^2 + \beta y'^2 &= 0,\\ \alpha x'^2 + \beta y' &= 0,\\ x'^2 &= 0. \end{aligned}$$

Analytically it means that we may introduce into the equation (**) new variables x', y' related with x and y by the formulas of the form

$$\begin{aligned} x' &= \alpha_{11}x + \alpha_{12}y + \alpha_{13},\\ y' &= \alpha_{21}x + \alpha_{22}y + \alpha_{23}, \end{aligned}$$

so that the equation (**) will take one of the above forms.

Whence it follows that if the projective transformation

$$\begin{aligned} x_1' &= \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3, \\ x_2' &= \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3, \\ x_3' &= x_3, \end{aligned}$$

is applied to the quadric curve (*), then we shall obtain one of the following curves:

$$\begin{aligned} \alpha x_1^2 + \beta x_2^2 + \gamma x_3^2 &= 0, \\ \alpha x_1^2 + \beta x_2^2 &= 0, \\ \alpha x_1^2 + \beta x_2 x_3 &= 0, \\ x_1^2 &= 0. \end{aligned}$$

As to these curves, they are readily carried into the curves (***) by a simple projective transformation. For example, in the first case we have to apply the projective transformation

$$x_1' = \sqrt{|\alpha|}\,x_1, \quad x_2' = \sqrt{|\beta|}\,x_2, \qquad x_3' = \sqrt{|\gamma|}\,x_3;$$

in the second

$$x_1' = \sqrt{|\alpha|}\,x_1; \quad x_2' = \sqrt{|\beta|}\,x_2; \qquad x_3' = x_3;$$

in the third

$$x_1' = \sqrt{|\alpha|}\,x_1; \quad x_2' = \frac{x_2 + x_3}{2}\sqrt{|\beta|}; \quad x_3' = \frac{x_2 - x_3}{2}\sqrt{|\beta|}.$$

A similar assertion may be proved for quadric surfaces in a space supplemented with infinite elements. Namely, *any quadric surface is projectively equivalent to one of the following surfaces*

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_4^2 &= 0, \\ x_1^2 + x_2^2 + x_3^2 - x_4^2 &= 0, \\ x_1^2 + x_2^2 - x_3^2 - x_4^2 &= 0, \\ x_1^2 + x_2^2 + x_3^2 &= 0, \\ x_1^2 + x_2^2 - x_3^2 &= 0, \\ x_1^2 + x_2^2 &= 0, \\ x_1^2 - x_2^2 &= 0, \\ x_1^2 &= 0. \end{aligned}$$

The proof is analogous to that given for curves.

EXERCISE

1. Find the projective transformations which map the curves

$$(a_1x_1 + a_2x_2 + a_3x_3)^2 \pm (b_1x_1 + b_2x_2 + b_3x_3)^2 = 0,$$
$$(a_1x_1 + a_2x_2 + a_3x_3)(b_1x_1 + b_2x_2 + b_3x_3) = 0.$$

into one of the canonical forms (***).

Sec. 9-9. The Pole and Polar

Introducing homogeneous coordinates into the formula (**) of Sec. 9-6 for the anharmonic ratio, we get

$$(ABCD) = \frac{\begin{vmatrix} x_{1A} & x_{1C} \\ x_{4A} & x_{4C} \end{vmatrix}}{\begin{vmatrix} x_{1B} & x_{1C} \\ x_{4B} & x_{4C} \end{vmatrix}} : \frac{\begin{vmatrix} x_{1A} & x_{1D} \\ x_{4A} & x_{4D} \end{vmatrix}}{\begin{vmatrix} x_{1B} & x_{1D} \\ x_{4B} & x_{4D} \end{vmatrix}} \qquad (*)$$

and respectively two other formulas with x_1 replaced by x_2 or x_3.

The anharmonic ratio of the points on a straight line in a space supplemented by infinite elements is defined by the formula (*). Independent of the proof given in Sec. 9-6, we can show that the anharmonic ratio thus defined is preserved under the projective transformation. We leave this to the student.

Suppose we have a quadric surface

$$2F = \sum_{i,\, j=1}^{4} a_{ij}x_ix_j = 0 \qquad (**)$$

and a point A (x_1', x_2', x_3', x_4'), not lying on the surface. Through the point A we draw an arbitrary straight line to intersect the surface (**) at points C and D. We then construct a point B, *harmonically separating* the points C and D from the point A, i.e. such that $(ABCD) = -1$.

The locus of the points thus constructed is called the *polar* of the point A. The point A is called the *pole* with reference to the polar.

We are going to form the equation of the polar. Let x_1, x_2, x_3, x_4 be the homogeneous coordinates of the point B. The coordinates $\bar{x}_i$ of any point of the line AB, different from A, may be represented in the form

$$\bar{x}_i = x_i + \lambda x_i' \quad (i = 1, 2, 3, 4). \qquad (***)$$

Indeed, the straight line AB is specified by two linear equations:

$$\sum a_i x_i = 0, \quad \sum b_i x_i = 0.$$

Since the rank of the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

is equal to two (the equations are independent), any solution of this system represents a linear combination of two independent variables:

$$\bar{x}_i = \mu x_i + \nu x_i' \quad (i = 1, 2, 3, 4).$$

If the point is different from A, then $\mu \neq 0$ and the coordinates $\bar{x}_i$ can be divided by μ to obtain the above representation.

We may convince ourselves by a direct check that the anharmonic ratio of four points A, B, $\lambda A + B$, $\mu A + B$ ($\xi A + B$ is a point with the coordinates $\varepsilon x_i + x_i'$)

$$(A, B, \lambda A + B, \mu A + B) = \frac{\lambda}{\mu}.$$

Whence it follows that the points C and D of intersection of the line AB with the quadric surface allows the representations

$$C = \lambda A + B, \quad D = -\lambda A + B$$

Substituting the coordinates of the points C and D into the equation of the surface, we get

$$\sum_{i,j} a_{ij} (\pm \lambda x_i + x_i')(\pm \lambda x_j + x_j') =$$

$$= \lambda^2 \sum_{i,j} a_{ij} x_i x_j \pm 2\lambda \sum_{i,j} a_{ij} x_i x_j' + \sum_{i,j} a_{ij} x_i' x_j' = 0.$$

Whence it follows:

$$\sum_{i,\,j} a_{ij}x_i x'_j = 0.$$

which is the *equation of the polar*. Hence, the polar represents a plane.

Let us here note two important properties of the polar:

1. *The polar of any point B belonging to the polar of point A passes through A.*

2. *If a point A moves along a straight line, then its polar turns about some straight line.*

Indeed, the equation of the polar of the point $B\ (x''_i)$

$$\sum_{i,\,j} a_{ij}x_i x''_j = 0$$

is satisfied by the coordinates of the point A, since

$$\sum_{i,\,j} a_{ij}x'_i x''_j = \sum_{i,\,j} a_{ij}x''_i x'_j$$

$$(a_{ij} = a_{ji}),$$

and

$$\sum_{i,\,j} a_{ij}x''_i x'_j = 0,$$

since B lies on the polar of the point A.

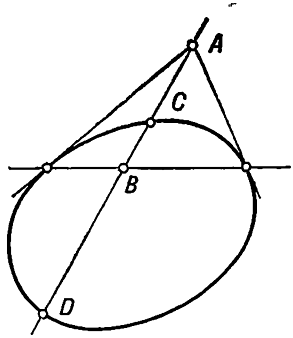


Fig. 93.

Suppose a point A moves in a straight line joining the points $A'\ (x'_i)$ and $A''\ (x''_i)$. The polar of any point of this line will be

$$\sum_{i,\,j} a_{ij}x_i (\lambda' x'_j + \lambda'' x''_j) = 0,$$

or

$$\lambda' \sum_{i,\,j} a_{ij}x_i x'_j + \lambda'' \sum_{i,\,j} a_{ij}x_i x''_j = 0.$$

Whence it is seen that the polar rotates about a straight line specified by the equations

$$\sum_{i,\,j} a_{ij}x_i x'_j = 0, \quad \sum_{i,\,j} a_{ij}x_i x''_j = 0.$$

The polar of the point A (x_1', x_2', x_3') with reference to a quadric curve is defined analogously (Fig. 93). It represents a straight line and is specified by the equation

$$\sum_{i, j=1}^{3} a_{ij}x_i x_j' = 0,$$

if the curve is specified by the equation

$$\sum_{i, j=1}^{3} a_{ij}x_i x_j = 0.$$

EXERCISES

1. Show that point C which together with an infinite point of the straight line AB harmonically separates the points A and B, is the mid-point of the line segment AB.

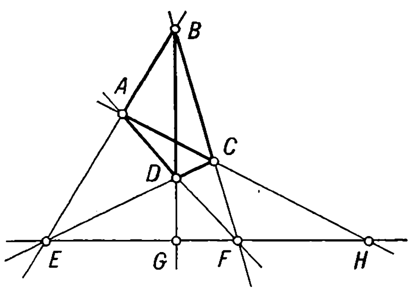


Fig. 94.

2. The *complete quadrilateral* is defined as a figure consisting of four points, no three of which are collinear, and six straight lines joining them pairwise (Fig. 94). Show that the pair of points G, H harmonically separates the pair of points E, F. (Make use of Exercise 1 and of the invariance of the anharmonic ratio under the projective transformation).

3. Justify the following method of constructing tangent lines to a conic section from an arbitrary point S (Fig. 95). Lines *1* and *2* are drawn arbitrarily, the rest of the lines in the order of numbers according to the figure.

4. How to draw a tangent line to a conic section at a given point using only a ruler?

5. Given a conic section and a straight line. How to construct the pole of the straight line with respect to the given conic section using only a ruler?

6. Let k be a conic section. We take an arbitrary straight line f and a point A on it. Construct the polar g of the

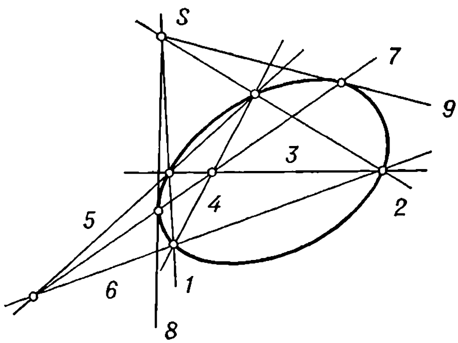


Fig. 95.

point A with respect to k. It will intersect f at point B. The polar h of the point B intersects the line g at point C and passes through the point A. In such a way we have constructed a triangle ABC whose sides are the polars of opposite vertices. This triangle is called a *self-polar* triangle.

Show that if the sides of the self-polar triangle are taken for the lines $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$, then the equation of the conic section k will have the form

$$\alpha x_1^2 + \beta x_2^2 + \gamma x_3^2 = 0.$$

7. Derive the properties of diameters and diametral planes from the properties of poles and polars.

8. Show that the polar of the focus of a conic section is the directrix.

Sec. 9-10. Tangential Coordinates

Any straight line contained in an extended plane may be uniquely associated with the ratio of three numbers $u_1 : u_2 : u_3$ which are the coefficients of its equation in homogeneous coordinates:

$$u_1x_1 + u_2x_2 + u_3x_3 = 0. \qquad (*)$$

The numbers u_1, u_2, u_3 will be called the *homogeneous coordinates of a straight line.* The homogeneous coordinates of a straight line are defined not uniquely. Namely, if u_1, u_2, u_3 are the homogeneous coordinates of a straight line, then ρu_1, ρu_2, ρu_3 ($\rho \neq 0$) will also be homogeneous coordinates of this line.

Let us find out the geometrical meaning of the equation

$$u_1x_1^0 + u_2x_2^0 + u_3x_3^0 = 0, \qquad (**)$$

in which u_1, u_2, u_3 are variables, and x_1^0, x_2^0, x_3^0 are fixed quantities.

To each solution u_1^0, u_2^0, u_3^0 of the equation (**) there corresponds a straight line

$$u_1^0x_1 + u_2^0x_2 + u_3^0x_3 = 0,$$

passing through the point (x_1^0, x_2^0, x_3^0). Conversely, the coordinates of any straight line passing through this point satisfy the equation (**). Hence, the equation (**) is satisfied by the coordinates of the straight lines forming a pencil with the centre (x_1^0, x_2^0, x_3^0), and only by them. This is why the equation (**) is called the *equation of a pencil.*

In case of a space we proceed in an analogous way, introducing the homogeneous coordinates of a plane u_1, u_2, u_3, u_4 as the coefficients of its equation in homogeneous coordinates.

For fixed x_i^0 and variable u_i the equation

$$u_1x_1^0 + u_2x_2^0 + u_3x_3^0 + u_4x_4^0 = 0$$

specifies a *bundle of planes* with centre (x_i^0).

The equation

$$\varphi\,(u_1,\ u_2,\ u_3) = 0,$$

which is satisfied by the homogeneous coordinates of tangent lines to a curve and only by them is called the *tangential equation of the curve.* Let us form the tangential equation of a non-degenerate quadric curve.

In Sec. 8-8 we obtained the equation of a tangent line to a quadric curve in Cartesian coordinates. Passing over to homogeneous coordinates, this equation is reduced to the following symmetric form:

$$x_1 F_{x_1'} + x_2 F_{x_2'} + x_3 F_{x_3'} = 0,$$

where

$$F_{x_1'} = a_{11}x_1' + a_{12}x_2' + a_{13}x_3',$$
$$F_{x_2'} = a_{21}x_1' + a_{22}x_2' + a_{23}x_3',$$
$$F_{x_3'} = a_{31}x_1' + a_{32}x_2' + a_{33}x_3'.$$

Whence it follows that the homogeneous coordinates of the tangent line at point (x_1', x_2', x_3') are

$$u_1 = F_{x_1'}, \quad u_2 = F_{x_2'}, \quad u_3 = F_{x_3'}.$$

Solving these equations with respect to x_1', x_2', x_3' (the determinant of the system is non-zero, since the curve is non-degenerate), we get for them the linear expressions in terms of u_1, u_2, u_3. Since the point (x_i') lies on the curve, its coordinates satisfy the equation of the curve. Substituting x_i' expressed in terms of u_i into the equation of the curve, we get the tangential equation of the curve. Obviously, it will be of the second degree and homogeneous with respect to the coordinates u_i:

$$2\Phi(u_1, u_2, u_3) = b_{11}u_1^2 + 2b_{12}u_1u_2 + \ldots + b_{33}u_3^2 = 0. \quad (***)$$

In connection with this a quadric curve is said to be a *curve of the second class.*

Let us find out *how to understand geometrically the totality of straight lines whose coordinates satisfy an arbitrary equation of the form* (***). As it has been shown, it may be a totality of tangent lines to a non-degenerate quadric curve. But this does not exhaust all the possibilities. For instance, the equation

$$(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3))(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3) = 0$$

specifies two pencils of straight lines with centres (α_i) and (β_i).

In Sec. 9-8 it was shown that any quadric curve

$$\sum_{i,\,j=1}^{3} a_{ij}x_ix_j = 0$$

can be carried by a projective transformation into the curve

$$\varepsilon_1 x_1^2 + \varepsilon_2 x_2^2 + \varepsilon_3 x_3^2 = 0,$$

where ε_i are numbers equal to $+1$, -1, or 0. Analytically it means that $\sum a_{ij}x_ix_j$ can always be represented in the form

$$\sum_{i=1}^{3} \varepsilon_i \left(\sum_{j=1}^{3} \alpha_{ij}x_j\right)^2,$$

the determinant formed from α_{ij} being non-zero.

Whence it follows that the equation (***) can always be reduced to the form

$$\sum_{i=1}^{3} \varepsilon_i \left(\sum_{j=1}^{3} \alpha_{ij}u_j\right)^2 = 0.$$

If all ε_i are non-zero, then this equation specifies tangent lines to a non-degenerate quadric curve. If one of the coefficients ε_i, say ε_3, is equal to zero, then the equation

$$\varepsilon_1 (\alpha_{11}u_1 + \alpha_{12}u_2 + \alpha_{13}u_3)^2 + \\ + \varepsilon_2 (\alpha_{21}u_1 + \alpha_{22}u_2 + \alpha_{23}u_3)^2 = 0$$

may be represented in the form of a product of two linear (with respect to u_i) factors (either real or complex):

$$(\beta_{11}u_1 + \beta_{12}u_2 + \beta_{13}u_3)(\beta_{21}u_1 + \beta_{22}u_2 + \beta_{23}u_3) = 0,$$

and the equation specifies two different pencils of straight lines. If both coefficients ε_i are zero, say ε_2 and ε_3, then the two pencils merge into one:

$$(\alpha_{11}u_1 + \alpha_{12}u_2 + \alpha_{13}u_3)^2 = 0.$$

Quadric surfaces in space are considered in just a similar way. We confine ourselves here to formulating the results obtained, omitting the work.

The tangential equation of a non-degenerate quadric surface has the form

$$\sum_{i,\ j=1}^{4} a_{ij}u_iu_j = 0.$$

The totality of planes whose homogeneous coordinates satisfy an arbitrary equation of the form

$$\sum_{i,\ j=1}^{4} a_{ij}u_iu_j = 0,$$

consists either of tangent planes to a non-degenerate quadric surfaces, or of planes passing through the tangent lines to a conic section, or of two bundles of planes which, in particular, may merge.

In conclusion, let us consider the so-called *correlation.* In an extended plane this is a linear transformation which carries a figure F consisting of points into a figure F' consisting of straight lines so that the coordinates of a straight line belonging to the figure F' are expressed in terms of coordinates of the corresponding point of the figure F according to the formulas

$$\begin{aligned} u_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ u_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ u_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3. \end{aligned} \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0.$$

This transformation allows a simple geometrical interpretation if $a_{ij} = a_{ji}$. Namely, it consists in associating the point $(x_1,\ x_2,\ x_3)$ with its polar with reference to a quadric curve specified by the equation

$$\sum a_{ij}x_ix_j = 0.$$

Whence it follows that points lying on a straight line go into straight lines passing through the point. This principal property of correlation takes place in the general case $(a_{ij} \neq a_{ji})$ as well.

Correlation in space is defined analogously. Each point A of the figure F is associated with the plane α of the figure F' whose coordinates are linearly expressed in terms of the coordinates of the point A. Correlation in space may be represented through the correspondence of poles and polars with reference to a quadric surface.

EXERCISES

1. *The anharmonic ratio of four straight lines of a pencil* is defined as an anharmonic ratio of four points of intersection of these lines with an arbitrary straight line not passing through the centre of the pencil. Show that this definition is invariant with respect to the secant straight line, and find the expression of the anharmonic ratio in terms of the homogeneous coordinates of the straight lines.

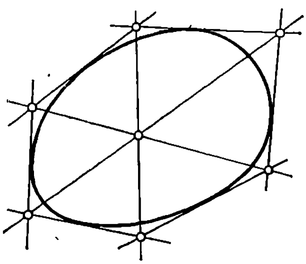


Fig. 96.

Show, in particular, that the anharmonic ratio of the straight lines (u_i), (v_i), $(u_i + \lambda v_i)$, and $(u_i + \mu v_i)$ is equal to λ/μ.

Show that under correlation the anharmonic ratio of four points of the figure F is equal to that of the corresponding straight lines (planes) belonging to the figure F'.

2. With the aid of Pascal's theorem (see Exercise 9 to Sec. 3-8) prove the following *Brianchon's theorem*: If a hexagon is circumscribed about a conic section, then the lines joining pairs of opposite vertices are concurrent (Fig. 96).

Answers to the Exercises, Hints and Solutions

Chapter 1

Sec 1-1

1. (a) The points of the xy-plane for which $|x| = a$ lie on two straight lines parallel to the y-axis at a distance a from it. (b) The points for which $|x| = |y|$ lie on the bisectors of the quadrants.

2. (a) The points of the xy-plane for which $|x| < a$ lie within the band between the straight lines parallel to the y-axis and situated at a distance a from it. (b) The points for which $|x| < a$, $|y| < b$ lie inside a rectangle with the centre at the origin and sides $2a$ and $2b$ parallel to the x- and y-axes.

3. The coordinates of the point symmetrical to the point $A(x, y)$ about the x-axis will be x and $-y$; the coordinates of the point symmetrical to the point $A(x, y)$ about the y-axis will be $-x$, y; and the coordinates of the point symmetrical to the point $A(x, y)$ about the origin will be $-x$, $-y$.

4. The coordinates of the point symmetrical to the point $A(x, y)$ about the bisector of the first (second) quadrant will be y, x $(-y, -x)$.

5. If the y-axis is taken for the x-axis and the x-axis for the y-axis, then the point $A(x, y)$ will have the abscissa y and the ordinate x.

6. If the origin is displaced to the point $A(x_0, y_0)$ without changing the direction of the coordinate axes, then the point $A(x, y)$ will have the abscissa $x - x_0$ and ordinate $y - y_0$.

7. If the diagonals of a square whose side is equal to $2a$ are taken for the coordinate axes, then the abscissa and ordinate of the mid-points of the sides of the square will be equal to $\pm a/\sqrt{2}$. The sign depends on the side taken. Four possible combinations of signs correspond to the four sides of the square.

8. For a point to be situated between two other points it is necessary that its abscissa (or ordinate) be enclosed between the abscissas (ordinates) of the two other points.

Sec. 1-2

1. Equating the distances of the required point $(x, 0)$ from the two given points, we find the equation for x:

$$(x_1 - x)^2 + (y_1 - 0)^2 = (x_2 - x)^2 + (y_2 - 0)^2,$$

or

$$2(x_2-x_1)x=y_2^2-y_1^2+x_2^2-x_1^2.$$

Whence we find x. In the particular case we have: $x=(b^2-a^2)/2b$.

2. Find the distance d between the points A and B. The third vertex (C) of the triangle is located at a distance d from the vertices A and B. The coordinates x, y of the vertex C are found from the two equations thus obtained. The problem has two solutions which correspond to the two triangles situated symmetrically about the straight line AB.

3. Knowing the coordinates of the vertices A and B of the square, we find the side of the square a as the distance between A and B. The third vertex C is determined from the condition that it is situated at a distance a from B and at a distance $a\sqrt{2}$ (the diagonal of the square) from A. The fourth vertex D is determined from the condition that it is found at a distance a from A and C and is different from B. The problem has two solutions.

4. If $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ are the vertices of a right-angled triangle with the right angle C, then the condition will be

$$(x_3-x_2)^2+(y_3-y_2)^2+(x_3-x_1)^2+(y_3-y_1)^2=$$
$$=(x_2-x_1)^2+(y_2-y_1)^2.$$

This condition represents a coordinate notation of the Pythagorean theorem for the triangle ABC.

5. If $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ are the vertices of the triangle, then the condition that the angle A of the triangle exceeds the angle B will be expressed as follows

$$(x_3-x_2)^2+(y_3-y_2)^2>(x_3-x_1)^2+(y_3-y_1)^2.$$

This follows from the fact that in any triangle the greater side is opposite the greater angle and, conversely, the greater angle is opposite the greater side.

6. Find the centre O of the circle circumscribed about the triangle ABC and compare the radius R of this circle with the distance from the centre to the vertex D. The quadrilateral will be inscribed in the circle if $OD=R$. The quadrilateral will not be inscribed in the circle if $OD\neq R$.

7. This is the "inequality of a triangle" for the points with the coordinates (a, b), (a_1, b_1), (a_2, b_2).

Sec. 1-3

1. Let for definiteness (x_1, y_1) and (x_3, y_3) be the opposite vertices of the parallelogram. Then the coordinates of the centre of the parallelogram will be

$$x_0=\frac{x_1+x_3}{2}, \qquad y_0=\frac{y_1+y_3}{2}.$$

The coordinates x, y of the fourth vertex are found from the equations

$$x_0 = \frac{x_2 + x}{2}, \qquad y_0 = \frac{y_2 + y}{2}.$$

2. The point of intersection of the medians divides each median in the ratio 2:1, as measured from the vertex. The coordinates of the point of intersection of the medians will be

$$x = \frac{x_1 + x_2 + x_3}{3}, \qquad y = \frac{y_1 + y_2 + y_3}{3}.$$

3. The mid-points of the sides of the triangle and any of its vertices taken together are the vertices of a parallelogram. Therefore the problem is reduced to Problem 1.

4. $x_i' = (1 - \lambda)\, x_0 + \lambda x_i$, $y_i' = (1 - \lambda)\, y_0 + \lambda y_i$, $i = 1, 2, 3$.

5. To solve this problem make use of the geometrical reasoning considered in connection with dividing a line segment in a given ratio (see Sec. 1-3).

6. Let (x_1, y_1) and (x_2, y_2) be the end-points of one line segment, and (x_3, y_3) and (x_4, y_4) the end-points of the other segment. If the segments intersect, then the coordinates of the point of intersection allow two methods of representation:

$$(1 - t)\, x_1 + t x_2 = (1 - t')\, x_3 + t' x_4,$$
$$(1 - t)\, y_1 + t y_2 = (1 - t')\, y_3 + t' y_4.$$

The segments intersect if the solutions of this system with respect to t and t' satisfy the conditions $0 \leqslant t,\ t' \leqslant 1$.

7. Use the method of mathematical induction.

Sec. 1-4

1. (1) If $a = 0$, the centre of the circle lies on the y-axis; (2) if $b = 0$, the centre of the circle lies on the x-axis; (3) if $c = 0$, the circle passes through the origin; (4) if $a = b = 0$, the centre of the circle is at the origin; (5) if $a = 0$ and $c = 0$, the circle touches the x-axis at the origin; (6) if $b = 0$ and $c = 0$ the circle touches the y-axis at the origin.

2. Pay attention to the fact that $(x - a)^2 + (y - b)^2$ is the square of the distance of the point (x, y) from the centre of the circle, and use the Pythagorean theorem as applied to the right-angled triangle in which one leg is a tangent to the circle, the other leg being the radius of the circle.

3. Make use of the fact that the power for external points is equal to the square of the length of the tangent, and for internal points to the square (taken with the minus sign) of half the length of the chord passing through the given point perpendicular to the diameter joining this point to the centre of the circle.

4. Suppose (x, y) is a point of the locus. Its distances from F_1 and F_2 are respectively equal to

$$\sqrt{(x-c)^2+y^2}, \qquad \sqrt{(x+c)^2+y^2}.$$

The equation of the locus of points:

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a.$$

To reduce this equation to the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

we have to transpose the first radical to the right-hand side of the equation and to square both members. We get

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2.$$

Leaving the radical in the right-hand member of the equation and transposing the remaining terms to the left-hand side, after obvious simplifications we get

$$cx-a^2=-a\sqrt{(x-c)^2+y^2}.$$

Squaring both members, after simple transformations we obtain

$$a^4-a^2c^2=a^2y^2+(a^2-c^2)\,x^2,$$

whence

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1, \qquad a^2-c^2=b^2.$$

5. The problem is solved analogously to the previous one, the initial equation being

$$\sqrt{(x-c)^2+y^2}-\sqrt{(x+c)^2+y^2}=\pm 2a.$$

6. The equation of the locus:

$$\sqrt{(y-p)^2+x^2}=y.$$

After squaring and simplifications the equation takes the form

$$-2py+p^2+x^2=0.$$

Sec. 1-5

1. The equation of the curve in implicit form:

$$(x-a)^2+(y-b)^2=R^2.$$

Whence it is seen that a and b are the coordinates of the centre, and R is the radius.

2. The equations of the curve:

$$x=\frac{\lambda a}{\lambda+\mu}\cos t, \qquad y=\frac{\mu a}{\lambda+\mu}\sin t.$$

For $\lambda=\mu$ the curve is a circle.

3. The equations of the curve:

$$x=a\cos t+h\sin t, \qquad y=b\sin t+h\cos t,$$

where a, b, h, and the parameter t have the values indicated in Fig. 14. To get these equations represent the abscissa x and ordinate y of the point on the curve in the form of an algebraic sum of the lengths of the projections of the links of the polygonal line $OABC$.

4. The equations of the curve:

$$x=R\left(\frac{s}{R}-\sin\frac{s}{R}\right), \qquad y=R\left(1-\cos\frac{s}{R}\right) \qquad (a\ cycloid)$$

The problem is solved like the preceding one. Here the polygonal line is $OTSA$.

5. Solving the equations

$$ax^2+bxy+cy^2+dx+ey=0, \qquad t=\frac{y}{x}$$

with respect to x and y, we get the equations of the curve in parametric form:

$$x=-\frac{d+et}{a+bt+ct^2}, \qquad y=-\frac{dt+et^2}{a+bt+ct^2}.$$

Sec. 1-6

1. The points of intersection of the circle with the x-axis are obtained by solving the system of equations

$$x^2+y^2+2ax+2by+c=0, \qquad y=0.$$

The circle does not intersect the x-axis if the roots of the equation

$$x^2+2ax+c=0$$

are imaginary. The circle intersects the x-axis at two points if the roots of this equation are real and different. The circle touches the x-axis if the roots coincide.

2. The circles intersect at two points if $R_1+R_2>d$, where R_1 and R_2 are the radii of the circles and d is the distance between their centres. R_1, R_2, and d can be expressed in terms of the coefficients of the equations of the circles. One can also find these conditions by solving the system formed from the equations of the given circles.

3. The points of intersection of the circles:

$$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right).$$

4. The point of intersection of the curves: (1, 0).

5. If the point (x, y) satisfies the equations of the curves, then the points $(-x, y)$ and $(x, -y)$ which are symmetrical to it about the coordinate axes also satisfy these equations. Therefore the points of intersection are situated symmetrically about the coordinate axes.

Chapter 2

Sec. 2-1

1. The equation may be written in an equivalent way

$$(ax + by + c)(ax + by - c) = 0.$$

It is satisfied only by the points of the straight lines $ax + by + c = 0$, $ax + by - c = 0$ and only by them.

2. See Sec. 1-6.

3. Suppose the straight lines intersect at some point (x_1, y_1). Then

$$ax_1 + by_1 + c = 0, \qquad Ax_1 + By_1 + C = 0.$$

Multiplying the first equation by A, and the second by a and subtracting termwise, we get $Ac - Ca = 0$. This equation together with $Ab - Ba = 0$ yield the following proportion

$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C},$$

from which it follows that both equations specify one and the same, but not different straight lines.

4. The radical axis of the circles

$$x^2 + y^2 + a_1x + b_1y + c_1 = 0, \qquad x^2 + y^2 + a_2x + b_2y + c_2 = 0$$

is represented by the equation

$$(a_1 - a_2)x + (b_1 - b_2)y + c_1 - c_2 = 0.$$

See Exercise 3 of Sec. 1-4.

5. If (x_1, y_1) and (x_2, y_2) are the two given points, then the uation of the locus will be

$$(x - x_1)^2 + (y - y_1)^2 - (x - x_2)^2 - (y - y_2)^2 = a.$$

This equation is linear with respect to x, y.

6. The point (x', y') lies on the ray passing through the point (x, y) and $\sqrt{x'^2 + y'^2} \cdot \sqrt{x^2 + y^2} = R^2$,

7. Let

$$x^2 + y^2 + ax + by + c = 0$$

be the equation of the given circle. Dividing it by $x^2 + y^2$, we get

$$1+\frac{ax}{x^2+y^2}+\frac{by}{x^2+y^2}+\frac{c}{x^2+y^2}=0.$$

Noting that

$$x' = \frac{R^2x}{x^2+y^2}, \qquad y' = \frac{R^2y}{x^2+y^2}, \qquad x'^2+y'^2=\frac{R^4}{x^2+y^2},$$

we obtain the equation of the transformed curve:

$$1+\frac{a}{R^2}x' + \frac{b}{R^2}y' + \frac{c}{R^4}(x'^2+y'^2)=0.$$

In the general case this is the equation of a circle. If $c = 0$, i.e. if the initial circle passes through the origin (the centre of inversion), a straight line is obtained.

8. The equation $ax + by + c = 0$ has an equivalent form

$$2(x_0-x_0')\,x+2(y_0-y_0')\,y+(x_0'^2+y_0'^2-x_0^2-y_0^2)=0,$$

when x_0' and y_0' are the coordinates of the point A^*. From the equivalence of the above equations there follows the proportion

$$\frac{2(x_0-x_0')}{a}=\frac{2(y_0-y_0')}{b}=\frac{x_0'^2+y_0'^2-x_0^2-y_0^2}{c}.$$

Whence we find x_0' and y_0'.

9. The fact that the determinant is equal to zero guarantees the existence of a non-trivial solution of the system

$$\left.\begin{array}{l} ax_1+by_1+c=0,\\ ax_2+by_2+c=0,\\ ax_3+by_3+c=0 \end{array}\right\}$$

with respect to a, b, c (a and b cannot be both equal to zero). The straight line passing through the given points has the equation $ax + by + c = 0$.

Sec. 2-2

1. The straight line intersects the positive semi-axis x if $c/a < 0$. It intersects the negative semi-axis x if $c/a > 0$.

2. The straight line does not intersect the first quadrant if a, b, $c \geqslant 0$, or a, b, $c \leqslant 0$.

3. If the point (x, y) satisfies the first equation, then the point $(x, -y)$ which is symmetrical to it about the x-axis satisfies the second equation.

4. If the point (x, y) satisfies the first equation, then the point $(-x, -y)$ which is symmetrical to it about the origin satisfies the second equation.

5. The straight line is parallel to the x-axis if λ satisfies the condition $a + \lambda a_1 = 0$. The straight line passes through the origin if $c + \lambda c_1 = 0$.

6. The straight line, together with the coordinate axes, bound an isosceles triangle if $|a| = |b|$.

7. $\left|\frac{c}{a}\right|$ and $\left|\frac{c}{b}\right|$ are the legs of the right-angled triangle.

8. The tangents are given by the equation of the form $x - \lambda = 0$ (or $y - \lambda = 0$). The quantity λ is determined by the condition that the equation $\lambda^2 + y^2 + 2a\lambda + 2by = 0$ ($x^2 + \lambda^2 + 2ax + 2b\lambda = 0$, respectively) has the unique solution with respect to y (x).

Sec. 2-3

1. Use the formula (*) given in Sec. 2-3.

2. The straight line forms with the x-axis an angle of $\pi/2 - \alpha$.

3. If the side of the triangle lies on the x-axis and its altitude on the positive semi-axis y, then the equations of its sides will be

$$y=0, \qquad y=\frac{\sqrt{3}}{2}+x\sqrt{3}, \qquad y=\frac{\sqrt{3}}{2}-x\sqrt{3}.$$

4. This is an isosceles triangle situated symmetrically about the bisector of the first quadrant.

5. Compare the angles formed by the straight lines with the x-axis. The required condition is $a/b = -a_1/b_1$.

6. Pass over to specifying the straight line by the equation $(x - b)c - (y - d)a = 0$.

7. Pass over to specifying the straight lines by the equations in implicit form.

8. The vertices of the quadrilateral are situated at the point $\left(\pm\frac{c}{a}, 0\right)$, $\left(0, \pm\frac{c}{b}\right)$.

Sec. 2-4

1. The straight lines are given by the equations of the form

$$\frac{x}{a}+\frac{y}{b}=1, \qquad \frac{x}{c}+\frac{y}{d}=1, \qquad |a|=|b|, \qquad |c|=|d|.$$

For these lines either $ad - bc = 0$, or $ac + bd = 0$.

2. Pass over to the implicit form of equations for the given lines. The parallelism condition is $\alpha_1\beta_2 - \beta_1\alpha_2 = 0$. The perpendicularity condition: $\alpha_1\alpha_2 + \beta_1\beta_2 = 0$.

3. The parallelism condition: $a\alpha + b\gamma = 0$. The perpendicularity condition: $a\gamma - b\alpha = 0$.

4. In the case of parallelism the parameter λ is determined from the condition $(a_1 + a_2\lambda) b - (b_1 + b_2\lambda) a = 0$. In the case of perpendicularity from the condition $(a_1 + a_2\lambda) a + (b_1 + b_2\lambda) b = 0$.

Sec. 2-5

1. When substituting the coordinates of the given point and the coordinates of any of the vertices in the left-hand side of the equation of the side of the triangle opposite this vertex, we must get expressions with the same sign. If at least for one vertex these expressions are of different signs, then it means that the point is situated outside the triangle.

2. We have

$$\left| \frac{ax+by+c_1}{\sqrt{a^2+b^2}} - \frac{ax+by+c_2}{\sqrt{a^2+b^2}} \right| = \frac{|c_1-c_2|}{\sqrt{a^2+b^2}}.$$

If $A(x, y)$ is a point on one of the straight lines, then the left-hand side of the identity represents the distance of this point from the other line, i.e. the distance between the lines.

3. The straight lines are represented by equations of the form

$$ax + by + c' = 0;$$

c' is determined from the equality $|c - c'| = \delta\sqrt{a^2 + b^2}$. See the previous problem.

4. The equation $(ax + by + c) \pm (a_1x + b_1y + c_1) = 0$ expresses the equality of distances of the point (x, y) from the given lines.

5. If the initial straight lines are given by the equations in the normal form

$$a_1x + b_1y + c_1 = 0, \qquad a_2x + b_2y + c_2 = 0,$$

then the equation of the locus of points will be

$$(a_1x + b_1y + c_1)\lambda \pm (a_2x + b_2y + c_2)\mu = 0.$$

It is linear and therefore the locus of points is a straight line.

Sec. 2-6

1. See Exercise 4 of Sec. 2-4.

2. $ax_1 + by_1 + c = -(ax_2 + by_2 + c)$ and $a(y_2 - y_1) - b(x_2 - x_1) = 0$. The first condition expresses the fact that the points are situated on different sides of the straight line and are equidistant from it. The second condition expresses the location of the points on a straight line which is perpendicular to the given one.

3. The straight line is given by the following equation

$$(x - x_0) - \lambda(y - y_0) = 0.$$

The parameter λ is determined from the condition

$$(x_1 - x_0) - \lambda (y_1 - y_0) = \pm\{(x_2 - x_0) - \lambda (y_2 - y_0)\}.$$

The choice of the sign depends on how the points are situated: on one side or on both sides of the line.

4. Subtracting the first row of the determinant from its other two rows and expanding the determinant, we get

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1).$$

The fact that the right-hand side is equal to zero means that the point (x_2, y_2) lies on the straight line

$$(x - x_1)(y_3 - y_1) - (y - y_1)(x_3 - x_1) = 0,$$

joining the points (x_1, y_1), (x_3, y_3).

Sec. 2-7

1.

$$x' = \pm \frac{ax + by + c_1}{\sqrt{a^2 + b^2}}, \qquad y' = \pm \frac{-bx + ay + c_2}{\sqrt{a^2 + b^2}}.$$

The choice of signs depends on the direction of the x'- and y'-axes.

2.

$$x' = \pm \frac{x + y}{\sqrt{2}}, \qquad y' = \pm \frac{x - y}{\sqrt{2}}.$$

The equation of the curve in the new coordinates:

$$2x'y' = a^2.$$

3. The point (x_0, y_0) has the same coordinates x_0, y_0 in the new system. Therefore, x_0, y_0 are obtained by solving the following system of equations:

$$x_0 = a_1 x_0 + b_1 y_0 + c_1, \qquad y_0 = a_2 x_0 + b_2 y_0 + c_2.$$

Chapter 3

Sec. 3-1

1. Taking the general equation of the circle in rectangular Cartesian coordinates, pass over to the polar coordinates, substituting $x = \rho \cos \theta$, $y = \rho \sin \theta$. To obtain the coordinates of the centre and the radius of the circle given by an equation in polar coordinates pass over to the rectangular Cartesian coordinates. We get

$$x^2 + y^2 + 2a (x \cos \alpha - y \sin \alpha) + b = 0,$$

The coordinates of the centre: $x = -a \cos \alpha$, $y = a \sin \alpha$. The radius: $R = \sqrt{a^2 - b}$. The coordinates of the centre in polar coordinates: $\rho = a$, $\theta = \pi - \alpha$.

2. The distance between the points A (ρ_1, θ_1) and B (ρ_2, θ_2) can be found by the law of cosines as applied to the triangle OAB. We get

$$|AB|^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_2 - \theta_1).$$

3. ρ_0 is the distance from the pole to the straight line, α is the angle between the polar axis and the perpendicular dropped from the pole onto the straight line.

4. The equation of the cardioid in polar coordinates:

$$\rho = R\,(1 - \cos\theta).$$

5. The equation of the lemniscate of Bernoulli:

$$\rho = a\sqrt{2\cos 2\theta},$$

where a is half the distance between the foci.

Sec. 3-2

1. The equation of the curve can be written in the form

$$\rho = \frac{\lambda p}{1 + \lambda \cos\theta'},$$

where

$$\lambda = \sqrt{a^2 + b^2}, \qquad p = c/\sqrt{a^2 + b^2}, \qquad \theta' = \theta + \alpha.$$

This form is taken by the equation of the curve if the polar axis is turned through an angle α. It is seen from this equation that the curve is a conic section.

2. Since the position of the polar axis is not specified, the equation of the ellipse has the form

$$\rho = \frac{\lambda p}{1 + \lambda \cos(\theta + \alpha)}.$$

Substituting the coordinates of the three given points of the ellipse in this equation, we get a system of equations for determining the unknowns α, λ and p.

3. Checked directly.

4. The equation of parabola:

$$\rho = \frac{e}{1 - \cos\theta}.$$

The inversion with respect to the pole of a polar coordinate system is given by the formulas

$$\theta' = \theta, \qquad \rho' = \frac{R^2}{\rho}.$$

This transformation of the parabola yields a curve represented by the equation

$$\rho' = (R^2/e)(1 - \cos\theta'),$$

which is a cardioid.

Sec. 3-3

1. That the curve is a conic section follows from the definition, since $(x - x_0)^2 + (y - y_0)^2$ is a square of the distance of the point (x, y) on the curve from the focus (x_0, y_0), and $ax + by + c$ is proportional to the distance of the point (x, y) from the directrix. The curve is an ellipse, parabola, or hyperbola depending on the magnitude of $k/\sqrt{a^2 + b^2} \lesseqgtr 1$.

2. The distance of a point on the conic section from the focus is proportional to its distance from the directrix. The distance from the directrix, as the distance from a straight line, is linearly expressed in terms of the coordinates of the point.

3. The problem of the intersection of a conic section with a straight line is reduced to solving a quadratic equation which cannot have more than two roots.

4. See Exercise 4 of Sec. 1-4.

5. See Exercise 5 of Sec. 1-4.

6. Let $A\,(x, y)$ be a point of the locus. Its distances from the centres of the given circles will be $|R \pm R_1|$, $|R \pm R_2|$, where R_1 and R_2 are the radii of the given circles, and R is the radius of the circle which touches them. The signs ($+$ or $-$) depend on whether the touching is internal or external. In any case either the sum, or the difference of the distances is constant. The locus of points represents an ellipse, a hyperbola, or a straight line. If one of the circles degenerates into a straight line, the locus of points is a parabola.

Sec. 3-4

1. Let

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

be the equation of the ellipse (a is the semi-major axis). Let us turn the xy-plane about the x-axis through an angle α, $\cos\alpha = \frac{b}{a}$. Then the circle $x'^2 + y'^2 = a^2$ in this plane is orthogonally projected into the given ellipse.

2. The equation of the hyperbola can be written in the form

$$\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 1.$$

$\frac{x}{a}+\frac{y}{b}=0$, $\frac{x}{a}-\frac{y}{b}=0$ are the equations of its asymptotes. For the point (x, y) of the hyperbola the factors in the left-hand member of its equation are proportional to the distances of this point from the asymptotes.

3. See the preceding problem.

4. Divide the segments AC and CD into n equal parts. Let A_m and C_m be the points of division with number m. Find the coordinates x, y of the point of intersection of the lines A_mB and C_mA, taking the line AB for the x-axis and the mid-point of the segment AB for the origin. Eliminating the parameter m/n, show that the points (x, y) satisfy the equation of the ellipse.

5. Make use of the hints to the preceding problem.

Sec. 3-5

1. The equation of the asymptotes of the hyperbola:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0.$$

The equation of the tangent line:

$$\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1.$$

Find the points of intersection of the tangent with the asymptotes. Eliminating y, we get a quadratic equation for x

$$x^2\left(\frac{1}{a^2}-\frac{b^2x_0^2}{y_0^2a^4}\right)+2\frac{b^2}{y_0^2}\frac{x_0}{a^2}x-\frac{b^2}{y_0^2}=0$$

or, noting that

$$\frac{1}{a^2}-\frac{b^2x_0^2}{y_0^2a^4}=-\frac{b^2}{y_0^2a^2},$$

we have

$$x^2-2x_0x+a^2=0.$$

Hence, it is seen that the product of the abscissas of the points of intersection of the tangent with the asymptotes is $x_1x_2=a^2$, and the area of the triangle

$$S=\frac{1}{2}\left(\frac{x_1}{\cos\alpha}\right)\left(\frac{x_2}{\cos\alpha}\right)\sin 2\alpha,$$

where α is the angle formed by the asymptotes with the x-axis,

2. Eliminating y from the equations, we get a quadratic equation for x. The condition of touching consists in that the equation has a multiple root, i.e. the discriminant of the equation is equal to zero. The discriminant of the equation represents a quadratic trinomial with respect to λ. For the tangent lines to intersect at

right angles in the point (x_0, y_0), it is necessary that $\lambda_2\lambda_1 = -1$. Therefore, equating the constant term of the reduced equation for λ to -1, we obtain the equation of the required locus. It turns out to be a circle.

3. Make use of the reasonings which helped to solve the preceding problem.

4. The equation of the pair of tangent lines to the ellipse:

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{\beta^2}{a^2}+\frac{\alpha^2}{b^2}\right)-\left(\frac{-\beta x}{a^2}+\frac{\alpha y}{b^2}\right)^2=0.$$

5. The abscissa of the mid-point of the segment with the ends at the point of intersection of the tangent line with the asymptotes is equal to x_0. Indeed, the abscissas of the points of intersection are the roots of the equation

$$x^2-2x_0x+a^2=0.$$

Hence, $(x_1+x_2)/2 = x_0$ (see the solution of Exercise 1 of this section).

Sec. 3-6

1. Find the coordinates of the focus according to the geometrical construction. Make sure that

$$c=\sqrt{a^2-b^2}.$$

2. The proof is analogous to the proof of the optical property of the ellipse.

3. The equation of the parabola can be written in the form

$$(x-c)^2+y^2=(ax+b)^2,$$

where c is the coordinate of the focus, and $ax+b=0$ is the equation of the directrix. Identifying this equation with the canonical equation $y^2-2px=0$, find c, a, and b.

4. See the solution of the previous problem.

5. Find the coordinates of the foci and make sure that they are independent of λ.

6. The equation with respect to λ

$$\frac{x_0^2}{a^2+\lambda}+\frac{y_0^2}{b^2+\lambda}=1$$

for any non-zero x_0, y_0 has two real roots λ_1, λ_2: $-a^2<\lambda_1<$ $<-b^2<\lambda_2$. To them there correspond an ellipse and a hyperbola passing through the point (x_0, y_0).

7. We have

$$\frac{x_0^2}{a^2+\lambda_1}+\frac{y_0^2}{b^2+\lambda_1}=1,\qquad \frac{x_0^2}{a^2+\lambda_2}+\frac{y_0^2}{b^2+\lambda_2}=1.$$

Subtracting these equalities termwise, we get

$$\frac{x_0^2}{(a^2+\lambda_1)(a^2+\lambda_2)}+\frac{y_0^2}{(b^2+\lambda_1)(b^2+\lambda_2)}=0,$$

which is the perpendicularity condition for the tangent lines

$$\frac{xx_0}{a^2+\lambda_1}+\frac{yy_0}{b^2+\lambda_1}=1, \qquad \frac{xx_0}{a^2+\lambda_2}+\frac{yy_0}{b^2+\lambda_2}=1.$$

Sec. 3-7

1. The slope of the diameter drawn through the point of tangency $k' = -b^2/a^2k$.

2. The direction of the diameter drawn through the point (x_0, y_0) is conjugate to the direction of the chord.

3. See Sec. 1-5. The conjugate directions correspond to the values of the parameter t differing by the angle $\pi/2$.

4. When projecting with a pencil of straight lines, parallel lines turn into parallel lines and the mid-point of a line segment turns into the mid-point of the segment. The conjugate diameters in a circle are perpendicular. The area S of a figure and the area of its projection $\overline{S}$ are related as follows: $\overline{S} = S \cos \alpha$, where α is the angle between the plane of the figure and the plane of projection.

5. See the hint to Exercise 4.

6. See the hint to Exercise 4.

7. Represent the ellipse as the projection of a circle.

8. Make use of the hint to Exercise 4. It is possible to inscribe a triangle in an ellipse so that the tangent line at each of its vertices is parallel to the opposite side. Here one of the vertices may be taken arbitrarily.

Sec. 3-8

1. Expand the left-hand member of the equation into a product of linear factors.

2. The curve is situated inside the parallelogram defined by the intersection of the two bands:

$$|ax+by+c| \leqslant \sqrt{k},$$
$$|\alpha x+\beta y+\gamma| \leqslant \sqrt{k}.$$

3. Take the bisectors of the angles formed by the straight lines $ax + by + c = 0$, $\alpha x + \beta y + \gamma = 0$ for the new coordinate axes.

4. The problem is reduced to the previous one by factoring the left-hand member of the equation into two linear co-factors.

5. See Sec. 1-6.

6. The second-order curve

$$ax^2 + bxy + cy^2 + dx + ey = 0$$

allows a parametric representation

$$x=-\frac{d+et}{a+bt+ct^2}, \quad y=-\frac{dt+et^2}{a+bt+ct^2} \quad \text{(Exercise 5 of Sec. 1-5)}$$

Then see Sec. 1-6.

7. See Sec. 1-6.

8. For the points A_i both terms of the left-hand member of the equation vanish. Take an arbitrary point A_0 on the curve γ at which $\alpha_{34}\alpha_{26}\alpha_{15} \neq 0$. Put

$$\lambda=\left.\frac{\alpha_{24}\alpha_{16}\alpha_{35}}{\alpha_{34}\alpha_{26}\alpha_{15}}\right|_{A_0}.$$

9. Make use of Exercise 8.

Chapter 4

Sec. 4-1

1. When all the vectors are turned through an angle of $2\pi/n$, their sum is also turned through the same angle. But as a result of this rotation the system of vectors transforms into itself. Therefore the sum is equal to zero.

2. For the point of intersection of the medians (O_0)

$$\overrightarrow{O_0A}+\overrightarrow{O_0B}+\overrightarrow{O_0C}=\mathbf{0}.$$

For any other point O this sum is equal to $3\overrightarrow{OO_0}$.

3. The equality expresses a well-known theorem of elementary geometry. The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.

4. Check directly.

5. The sum of vectors emanating from a point of the plane α and directed in one half-space relative to this plane is a vector directed in the same half-space.

6. The system of vectors r'_{mn} with a common origin at point (0, 0) and the terminal points at point x $(m\delta, n\delta)$ is situated symmetrically about the point (0, 0).

7. For the origin the sum of the vectors is equal to zero due to the symmetry of the system of the vectors. For any other point O' it is equal to $n\overrightarrow{O'O}$, where n is the number of vectors.

Sec. 4-2

1. If $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are non-zero and non-parallel vectors, then the sum $\lambda_1 r_1+\lambda_2 r_2$ is equal to zero if and only if $\lambda_1=0$, $\lambda_2=0$.

2. Represent one of the vectors in the form of a linear combination of two others.

3. Construct a triangle in which one side is represented by the vector $\boldsymbol{r}$, and two other sides are parallel to the vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

Sec. 4-3

1. Multiply the vector equality scalarly by the vector $\overrightarrow{A_1A_2}$ and the vector perpendicular to it.

2. $\lambda^2 a^2 + 2\lambda\mu\,(ab) + \mu^2 b^2 \equiv (\lambda a + \mu b)^2$.

3. Any three vectors parallel to one plane are linearly dependent, i.e. there exist numbers λ_1, λ_2, λ_3 not all equal to zero, such that

$$\lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3 = 0 \quad \text{(see Exercise 3 of Sec. 4-2).}$$

Multiplying this equality scalarly by r_1, r_2, r_3, we get

$$\left.\begin{array}{l} \lambda_1 (r_1 r_1) + \lambda_2 (r_1 r_2) + \lambda_3 (r_1 r_3) = 0, \\ \lambda_1 (r_2 r_1) + \lambda_2 (r_2 r_2) + \lambda_3 (r_2 r_3) = 0, \\ \lambda_1 (r_3 r_1) + \lambda_2 (r_3 r_2) + \lambda_3 (r_3 r_3) = 0. \end{array}\right\}$$

This system of equations with respect to λ_1, λ_2, λ_3 has a non-trivial solution. Therefore the determinant of the system is equal to zero.

4. See the hint to Exercise 3.

5. See the hint to Exercise 3.

6. See Exercise 5.

Sec. 4-4

1. The vectors $a \times b$ and c are parallel.

2. The vectors $(a \times b) \times c$ and $b\,(ac)$ are equal by absolute value and are in the same direction.

3. Represent the vector a in the form of a sum of two vectors one of which is parallel and the other is perpendicular to c.

4. Use the results of the three previous exercises.

5. If a, b, c are vectors with a common origin at the vertex of the pyramid and whose terminal points are at the vertices of its base, then

$$S = \frac{1}{2} \,|(a - b) \times (a - c)|.$$

Sec. 4-5

1. $(a \times b) \times c = b\,(ac) - a\,(bc)$. See Exercise 4 of Sec. 4-4.

2. Take for a, b, c the vectors with a common origin at the centre of the sphere and whose terminal points are at the vertices of the spherical triangle.

3. Use the formula of Exercise 4 of Sec. 4-4.

4. Use the identity

$$(a \times b) \times (c \times d) + (c \times d) \times (a \times b) = 0.$$

5. The vector r allows the representation $r = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$. Determine λ_1, λ_2, λ_3, multiplying scalarly this equality by the vectors $e_1 \times e_2$, $e_2 \times e_3$, $e_3 \times e_1$.

6. See Exercise 5.
7. The vectors $e_1 \times e_2$, $e_2 \times e_3$, $e_3 \times e_1$ are linearly independent. Represent the vector $\boldsymbol{r}$ in the form

$$r = \lambda_1 (e_2 \times e_3) + \lambda_2 (e_3 \times e_1) + \lambda_3 (e_1 \times e_2).$$

Multiplying this equality scalarly by e_1, e_2, e_3 find λ_1, λ_2, λ_3.
8. Represent the solution in the form

$$x = \lambda_1 (\boldsymbol{b} \times \boldsymbol{c}) + \lambda_2 (\boldsymbol{c} \times \boldsymbol{a}) + \lambda_3 (\boldsymbol{a} \times \boldsymbol{b}).$$

Multiplying this equality scalarly by $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ find λ_1, λ_2, λ_3.

Sec. 4-6

1. See Exercise 5 of Sec. 4-5.
2. See Exercise 7 of Sec. 4-5.
4. Use the identity of Exercise 3 of Sec. 4-5.
5. Use the identity of Exercise 3.
6. Use the identities of Exercises 4 and 3.

Chapter 5

Sec. 5-1

1. (a) The points for which $x = 0$ lie in the coordinate plane passing through the y- and z-axes; (d) the points for which $x = 0$ and $y = 0$ lie on the z-axis.
2. Eight points.
3. The points are situated inside a parallelepiped bounded by the planes $x = \pm a$, $y = \pm b$, $z = \pm c$.
4. The coordinates of the vertices of the parallelepiped $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, where ε_1, ε_2, ε_3 take on the values 1 or 0.
5. If the point $A\ (x, y, z)$ when rotated through an angle of $\pi/2$ is carried into the point $A'\ (x', y', z')$, then

$$(\overrightarrow{OA} \cdot e)\, e + \overrightarrow{OA} \times e = \overrightarrow{OA}', \qquad e = \overrightarrow{OA}_0 / |\overrightarrow{OA}_0|.$$

6. See Exercise 5. In the case of an arbitrary angle of rotation α

$$(\overrightarrow{OA} \cdot e)\, e + \overrightarrow{OA} \cos \alpha + (\overrightarrow{OA} \times e) \sin \alpha = \overrightarrow{OA}'.$$

Sec. 5-2

1. The square of the distance between the points $A\ (x, y, z)$ and $A'\ (x', y', z')$

$$(AA')^2 = \{(x - x')\, e_x + (y - y')\, e_y + (z - z')\, e_z\}^2 =$$
$$= (x - x')^2 + (y - y')^2 + (z - z')^2 +$$
$$+ 2\, (x - x')\, (y - y') \cos \gamma + 2\, (y - y')\, (z - z') \cos \alpha +$$
$$+ 2\, (z - z')\, (x - x') \cos \beta.$$

2. The centre of the circumscribed sphere is equidistant from the vertices of the tetrahedron.

3. The coordinates of the mid-points of the line segments joining the mid-points of any two opposite edges of the tetrahedron:

$$x=\frac{x_1+x_2+x_3+x_4}{4},$$

$$y=\frac{y_1+y_2+y_3+y_4}{4},$$

$$z=\frac{z_1+z_2+z_3+z_4}{4}.$$

4. The straight lines intersect at the centroid of the tetrahedron.

5. A point with the coordinates x, y, z lies on one side with any vertex A_i relative to the plane containing the opposite face.

6. If the vertices of the triangle are A_1, A_2, A_3, then

$$S=\frac{1}{2}\,|\,\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}\,|.$$

7. Show that by means of elementary transformations the determinants

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix}, \qquad \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

can be converted into each other.

8. See Exercise 7.

Sec. 5-3

1. The equation can be written in the equivalent form

$$(x+a)^2+(y+b)^2+(z+c)^2=a^2+b^2+c^2-d;$$

the coordinates of the centre: $-a$, $-b$, $-c$; the radius of the sphere: $(a^2+b^2+c^2-d)^{1/2}$.

2. The coordinates of any point satisfying the equations $f_1=0$ and $f_2=0$ also satisfy the equation

$$\lambda_1 f_1+\lambda_2 f_2=0.$$

3. The surface is generated by straight lines which are parallel to the z-axis and intersect the xy-plane along the curve specified by the equation $\varphi(x, y)=0$.

4. The equation of the cone:

$$z^2=(x^2+y^2+z^2)\cos^2\alpha,$$

where α is the angle between the generatrix and the axis.

5. The curves γ_1 and γ_2 can be specified by equations in parametric form:

$$\gamma_1 : x = u, \quad y = 0, \quad z = au^2,$$
$$\gamma_2 : x = 0, \quad y = v, \quad z = bv^2.$$

The surface generated by the mid-point of the line segment with the end-points on γ_1 and γ_2 is represented by the equations:

$$x = \frac{u}{2}, \qquad y = \frac{v}{2}, \qquad z = \frac{au^2 + bv^2}{2}.$$

6. The curves can be represented parametrically:

$$\gamma_1 : x = u, \quad y = a, \quad z = f(u);$$
$$\gamma_2 : x = u, \quad y = b, \quad z = \varphi(u).$$

The equation of the surface:

$$x = u, \quad y = (1 - v)\, a + vb, \quad z = (1 - v)\, f(u) + v\varphi(u).$$

7. $\sqrt{x^2 + y^2}$ is the distance of the point on the surface from the z-axis.

8. This is a cylindrical surface with the generatrix parallel to the z-axis. It passes through the curve, since the equation of the surface can be written in the equivalent form

$$(f(z) - z) - (\varphi(y) - z) = 0.$$

Hence it is seen that a point which satisfies the equations $z = f(x)$ and $z = \varphi(y)$, satisfies also the equation $f(x) - \varphi(y) = 0$.

Sec. 5-4

1.

$$x' = a_1 x + b_1 y + c_1,$$
$$y' = a_2 x + b_2 y + c_2,$$
$$z' = z.$$

2.

$$\cos\alpha = \frac{a_{23}}{\sqrt{a_{22}a_{33}}}, \qquad \cos\beta = \frac{a_{31}}{\sqrt{a_{33}a_{11}}}, \qquad \cos\gamma = \frac{a_{12}}{\sqrt{a_{11}a_{22}}}.$$

See Exercise 1 of Sec. 5-2.

3. See Exercises 5 and 7 of Sec. 4-5.

Chapter 6

Sec. 6-1

1. An arbitrary point of the plane is equidistant from the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

2. The system of equations

$$\left.\begin{aligned} &ax+by+cz+d_1=0, \\ &ax+by+cz+d_2=0, \quad d_1 \neq d_2, \end{aligned}\right\}$$

has no solution. Therefore the planes do not intersect.

3. The locus consists of two planes

$$ax + by + cz + d \pm (\alpha x + \beta y + \gamma z + \delta) = 0.$$

4. Any solution of the system of equations

$$\left.\begin{aligned} &f(x, y, z)+a_1x+b_1y+c_1z+d_1=0, \\ &f(x, y, z)+a_2x+b_2y+c_2z+d_2=0 \end{aligned}\right\}$$

is a solution of the equation

$$(a_1x + b_1y + c_1z + d_1) - (a_2x + b_2y + c_2z + d_2) = 0.$$

5. For $k \neq \lambda d + \mu\delta$ the system of equations specifying the planes is incompatible.

6. See Exercise 4.

7. See Exercises 6 and 7 of Sec. 2-1.

8. The equation of any plane passing through the line of intersection of the two given planes is written in the form

$$(ax + by + cz + d)(\alpha x_0 + \beta y_0 + \gamma z_0 + \delta) - \\ - (\alpha x + \beta y + \gamma z + \delta)(ax_0 + by_0 + cz_0 + d) = 0,$$

where (x_0, y_0, z_0) is an arbitrary point outside the line of intersection.

9. See Exercise 8 of Sec. 5-2.

Sec. 6-2

1. The plane intersects the positive semi-axis x (y, z) if $d/a < 0$ ($d/b < 0$, $d/c < 0$, respectively).

2. The volume of the tetrahedron:

$$V = \frac{1}{6}\left|\frac{d^3}{abc}\right|.$$

3. The set of points in space satisfying the condition

$$|x| + |y| + |z| < a,$$

is the intersection (the common portion) of the half-spaces specified by the inequalities

$$\pm x \pm y \pm z < a.$$

4. The plane symmetrical to the plane σ about the xy-plane is specified by the equation

$$ax + by - cz + d = 0.$$

The plane symmetrical to the plane σ about the origin is given by the equation

$$-ax - by - cz + d = 0.$$

5. A plane parallel to the z-axis has no z in its equation. Consequently, the parameter λ is determined from the condition $c + \gamma\lambda = 0$.

6. The parameters λ and μ are determined from the conditions

$$a_1 + \lambda a_2 + \mu a_3 = 0, \qquad b_1 + \lambda b_2 + \mu b_3 = 0.$$

Sec. 6-3

1. The distance between the planes

$$\delta = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

See Exercise 2 of Sec. 2-5.

2. $\delta = \dfrac{|d|}{\sqrt{a^2 + b^2}}$.

3. If the planes are given by equations in the normal form

$$a_1x + b_1y + c_1z + d_1 = 0, \qquad a_2x + b_2y + c_2z + d_2 = 0,$$

then the locus of points is represented by the equation

$$a_1x + b_1y + c_1z + d_1 \pm \lambda\,(a_2x + b_2y + c_2z + d_2) = 0,$$

hence, it consists of two planes.

4. See Exercise 1.

5. Pass over to the normal form of the equation of a plane.

6. See Exercise 1 of Sec. 2-5.

7. If the equations of the planes are reduced to the normal form, then

$$\pm x' = a_1x + b_1y + c_1z + d_1,$$
$$\pm y' = a_2x + b_2y + c_2z + d_2,$$
$$\pm z' = a_3x + b_3y + c_3z + d_3.$$

Sec. 6-4

1. The vector (a, b, c) is perpendicular to the plane. The angle α formed by the plane with the x-axis is determined from the condition

$$\sin\alpha = \frac{|a|}{\sqrt{a^2 + b^2 + c^2}}.$$

2. The angle formed by the plane with the xy-plane is determined from the condition

$$\cos\alpha=\frac{1}{\sqrt{1+p^2+q^2}}.$$

3. See Exercise 2.
4. The plane intersects the x- and y-axes at equal angles if $|a|=|b|$.
5. The plane specified by the equation $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ passes through the point (x_0, y_0, z_0). The parallelism condition of planes is fulfilled.
6. The plane given by the equation

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0,$$

passes through the point (x_0, y_0, z_0) and is perpendicular to the vector

$$\left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right).$$

7. The parameters λ and μ obey the condition

$$(\lambda a_1+\mu a_2)\,a+(\lambda b_1+\mu b_2)\,b+(\lambda c_1+\mu c_2)\,c=0.$$

8. For any vector $\mathbf{n}$ (a, b, c) there can be found a plane with the normal $\mathbf{n}$ in the pencil of planes. To this end it is necessary to take the parameters $\lambda_1, \lambda_2, \lambda_3$ satisfying the conditions

$$\frac{\lambda_1 a_1+\lambda_2 a_2+\lambda_3 a_3}{a}=\frac{\lambda_1 b_1+\lambda_2 b_2+\lambda_3 b_3}{b}=$$
$$=\frac{\lambda_1 c_1+\lambda_2 c_2+\lambda_3 c_3}{c}.$$

Sec. 6-5

1. The straight line intersects the x-axis (y- or z-axes) if $\frac{y_0}{l}=\frac{z_0}{m}\left(\frac{x_0}{k}=\frac{z_0}{m},\ \frac{x_0}{k}=\frac{y_0}{l}, \text{ respectively}\right)$. The line is parallel in the plane xy (yz, or zx, correspondingly) if $m=0$ ($k=0$, $l=0$, respectively).
2. Form the equation of the locus of points, taking the equations of the planes in the normal form.
3. The locus of points equidistant from two vertices of the triangle is a plane. The required locus is the intersection of two planes, i.e. a straight line.
4. The straight line specified by the intersection of the plane $y=\lambda$, $z=a\lambda x$ lies on the surface, since the points of this line satisfy the equation of the surface. The straight line specified by the equations $x=\mu$, $z=a\mu y$ also lies on the surface.

5. The fact that the determinant is equal to zero is the condition of compatibility of the system of equations

$$\left.\begin{array}{l} a_1x+b_1y+c_1z+d_1=0, \\ a_2x+b_2y+c_2z+d_2=0, \\ a_3x+b_3y+c_3z+d_3=0, \\ a_4x+b_4y+c_4z+d_4=0. \end{array}\right\}$$

This system is compatible, since the straight lines intersect.

Sec. 6-6

1. The vanishing of the determinant means that the vectors

$$(x'-x'',\quad y'-y'',\quad z'-z''),\quad (k',\ l',\ m'),\quad (k'',\ l'',\ m'')$$

are parallel to the plane. Hence, the straight lines are either parallel, or intersect.

2. If A' and A'' are the points on the skew lines and e', e'' are the vectors indicating the directions of the lines, then the distance between the skew lines is

$$\delta=\pm\frac{e'\times e''}{|e'\times e''|}\overrightarrow{A'A''}=\pm\frac{(e'e''\overrightarrow{A'A''})}{|e'\times e''|}.$$

3. The vector indicating the direction of the line given by the equations

$$\left.\begin{array}{l} a_1x+b_1y+c_1z+d_1=0, \\ a_2x+b_2y+c_2z+d_2=0, \end{array}\right\}$$

has the coordinates

$$\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},\quad \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix},\quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

4. See Exercise 3.

5. The equation of the conical surface:

$$\frac{[(x-x_0)a+(y-y_0)b+(z-z_0)c]^2}{a^2+b^2+c^2}=$$
$$=[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]\sin^2\alpha.$$

6. See Exercise 3.

7. Let A (x, y, z) be a point on the conical surface different from the vertex. Find the coordinates of the point of intersection of the element passing through the point A with the plane $ax+by+$ $+cz+d=0$. Substituting these coordinates in the equation of the sphere $x^2+y^2+z^2=2Rz$, we obtain the equation of the required conical surface. The intersection of the conical surface with the xy-plane is a circle.

8. See Exercise 7.

Sec. 6-7

1. If the straight lines are given by the equations

$$\frac{x-x'}{k'}=\frac{y-y'}{l'}=\frac{z-z'}{m'}, \qquad \frac{x-x''}{k''}=\frac{y-y''}{l''}=\frac{z-z''}{m''},$$

then the plane equidistant from them passes through the point with the coordinates

$$\frac{x'+x''}{2}, \qquad \frac{y'+y''}{2}, \qquad \frac{z'+z''}{2}$$

parallel to the vectors (k', l', m'), (k'', l'', m'').

2. The plane specified by the equation

$$\frac{a_1x+b_1y+c_1z+d_1}{a_1x_0+b_1y_0+c_1z_0+d_1}=\frac{a_2x+b_2y+c_2z+d_2}{a_2x_0+b_2y_0+c_2z_0+d_2},$$

passes through the given line and the point (x_0, y_0, z_0), not lying on this line.

3. The vector $(x'-x_0, y'-y_0, z'-z_0)\times(k, l, m)$ is perpendicular to the required plane.

4. Any straight line intersecting the two given lines can be represented as the intersection of two planes one of which passes through the first line and the other through the second.

5. The surface specified by the equation of the form $\varphi\left(\frac{x}{z}, \frac{y}{z}\right)=0$ is generated by the straight lines passing through the origin, since, along with the point (x, y, z), any point $(\lambda x, \lambda y, \lambda z)$ satisfies the equation. The surface intersects the plane $z = 1$ along the curve $\varphi(x, y) = 0$.

Chapter 7

Sec. 7-2

1. The surface $z = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a$ represents an elliptic paraboloid (hyperbolic paraboloid, parabolic cylinder).

2. The left-hand member of the equation is factored into a product of two linear co-factors.

3. The equation thus obtained is satisfied by the coordinates of the points on the curve along which the plane intersects the surface.

4. See Exercise 3.

5. Form the equation of the conical surface, taking the given point for the origin and the plane containing the curve for the plane z = const. See Exercise 5 of Sec. 6-7.

6. The coordinates of the common points of the surfaces satisfy the equation $f(x, y, z)\,\varphi(x_0, y_0, z_0) - \varphi(x, y, z)\,f(x_0, y_0, z_0) = 0$. The point (x_0, y_0, z_0) also satisfies it.
7. The equation of the surface is a corollary of the equations of the straight lines. It is obtained by eliminating the parameter λ from them.
8. A second-degree surface. See Exercise 7.
9. $x^2+y^2=\left(\frac{z-b}{a}\right)^2+\left(\frac{z-d}{c}\right)^2.$

Sec. 7-3

1. The foci are found on the z-axis at a distance $\sqrt{c^2-a^2}$ from the origin.
2. The intersection of the ellipsoid with the plane is at the same time the intersection of these planes with the sphere $x^2 + y^2 + z^2 + \mu = 0$.
3. The required points in space are situated inside the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

4. Eliminate the parameters u, v and pass over to the equation of the surface represented in the implicit form.
5. The surface in question is an ellipsoid. Make use of the boundedness of the surface.

Sec. 7-4

1. See Exercise 2 of Sec. 7-3.
2. See Exercise 5 of Sec. 3-6.

Sec. 7-5

1. The focus of the paraboloid coincides with the focus of the parabola $z = y^2/a^2$ in the yz-plane.
2. Consider the projection of the section on the xy-plane. See Exercise 2 of Sec. 7-3.

Sec. 7-6

1. Make use of the fact that the vectors (λ, μ, ν) and (x, y, z) form an angle α.
2. If $\bar{A}$ is the projection of the point $A\,(x, y, z)$ on the line $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$, then $(O\bar{A})^2 + R^2 = (OA)^2$. Express $|O\bar{A}|$ in terms of the triple scalar product of the vectors (λ, μ, ν) and (x, y, z).

Sec. 7-7

1. Consider the projection of the line of intersection on the xy-plane. See Exercise 3 of Sec. 7-2.
2. The first family: $x = \lambda$, $z = a\lambda y$. The second family: $y = \mu$, $z = a\mu x$.
3. A hyperbolic paraboloid.

Chapter 8

Sec. 8-1

1. If the linear co-factors are independent, then introduce new variables

$$x_1' = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4,$$
$$x_2' = b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4,$$
$$x_3' = x_3, \quad x_4' = x_4.$$

The discriminant of the transformed form is obviously equal to zero.

2. In the case of independent linear forms

$$\sum a_ix_i, \quad \sum b_ix_i, \quad \sum c_ix_i, \quad \sum d_ix_i$$

introduce new variables

$$x_1' = \sum a_ix_i, \quad x_2' = \sum b_ix_i, \quad x_3' = \sum c_ix_i, \quad x_4' = \sum d_ix_i.$$

Sec. 8-2

1. $I_1 = a + c$, $I_2 = ac - b^2$, $I_3 = 0$, $I_4 = ac\gamma^2$.
2. Introduce new coordinates, taking the plane $ax + by + cz = 0$ for the coordinate plane.

Sec. 8-3

1. The condition of decomposition of the curve: $| a_{ij} + \lambda b_{ij} | = 0$. For the points of intersection of the curves both terms of the left-hand member of the equation of the curve are equal to zero.
2. Choose the parameter λ so that the equation

$$a_0y^2 + a_1xy + a_2x^2 + a_3x + a_4 + \lambda (y - x^2) = 0$$

is decomposed into a product of two linear co-factors.

3. $\alpha = \dfrac{I_1}{2\sqrt{-I_2}}, \qquad \beta = \dfrac{I_3}{2(\sqrt{-I_2})^3}.$
4. $\alpha = \sqrt{1 - \dfrac{4I_2}{I_1^2}}, \qquad \delta = \dfrac{2I_3}{I_1I_2}.$

Sec. 8-4

1. Take the bisector planes for the planes

$$ax + by + cz + d = 0, \quad a_1x + b_1y + c_1z + d_1 = 0$$

for the coordinate planes.

2. Find I_4 for the canonical equations of the surfaces.
3. Find I_4 and I_3 for the canonical equations of the surfaces.

Sec. 8-5

1. The origin satisfies the equations $F_{x'} = F_{y'} = 0$. Therefore $a'_{13} = a'_{23} = 0$. The constant term of the equation a'_{33} is determined by comparing the invariants $I'_3 = I_3$. The coefficients a'_{ij} for $i, j \leqslant 2$ are equal to a_{ij}.
2. See Exercise 1.

Sec. 8-6

1. Take the plane $ax + by + cz = 0$ for a coordinate plane. The axis of the cone is perpendicular to this plane.
2. The diameters of the parabola are parallel to the straight line $ax + by + c = 0$. The axis of the parabola is conjugate to the direction $a : b$.

Sec. 8-7

1. The asymptotes: $ax + by + c = 0$, $a_1x + b_1y + c_1 = 0$.
2. The asymptotes: $\lambda(ax + by + c) \pm \sqrt{-\lambda\mu}(a_1x + b_1y + c_1) = 0$.

Sec. 8-8

1. Use the canonical form of the equation of the surface.
2. See Sec. 3-5.
3. See Exercise 2.
4. See Exercise 4 of Sec. 3-5.
5. These straight lines are rectilinear generatrices.
6. See Exercise 7 of Sec. 3-6.

Chapter 9

Sec. 9-1

1. The formulas of the orthogonal transformation under which the xy-plane goes into itself:

$$x' = a_{11}x + a_{12}y + a_{14}, \quad y' = a_{21}x + a_{22}y + a_{24}, \quad z' = z;$$

$$a_{11}^2 + a_{21}^2 = 1, \quad a_{12}^2 + a_{22}^2 = 1, \quad a_{11}a_{12} + a_{21}a_{22} = 0.$$

2. The coefficients a_{11}, a_{21}, a_{31} are proportional to λ, μ, ν; $a_{14} = a_{24} = a_{34} = 0$.

Sec. 9-2

1. The formulas of the affine transformation:

$$x' = x_1 + (x_2 - x_1)\,x + (x_3 - x_1)\,y + (x_4 - x_1)\,z,$$
$$y' = y_1 + (y_2 - y_1)\,x + (y_3 - y_1)\,y + (y_4 - y_1)\,z,$$
$$z' = z_1 + (z_2 - z_1)\,x + (z_3 - z_1)\,y + (z_4 - z_1)\,z.$$

2. Solve the equations $\lambda x = ax' + by' + c$, $\mu y = a_1x' + b_1y' + c_1$ with respect to x' and y'.

Sec. 9-3

1. The xy-plane goes into the plane

$$x' = a_{11}u + a_{12}v + a_{14}, \quad y' = a_{21}u + a_{22}v + a_{24},$$
$$z' = a_{31}u + a_{32}v + a_{34};$$

u, v are parameters.

2. The x-axis goes into the line $x' = a_{11}t + a_{14}$, $y' = a_{21}t + a_{24}$, $z' = a_{31}t + a_{34}$; t parameter.

Sec. 9-4

1. Any three points not lying on a straight line can be carried by the affine transformation into any three points not lying on a straight line.

2. It is sufficient to carry three vertices of the parallelogram into three vertices of a square which is always possible. Not any quadrilateral can be transformed into a square by the affine transformation. The opposite sides of a quadrilateral must be parallel.

3. The system of equations must be compatible

$$x = a_{11}x + a_{12}y + a_{13}z + a_{14}, \quad y = a_{21}x + a_{22}y + a_{23}z + a_{24},$$
$$z = a_{31}x + a_{32}y + a_{33}z + a_{34}.$$

Sec. 9-5

1. The affine transformation preserves the conjugacy property.

2. The coefficients of stretching (compression) are equal to the semi-axes of the ellipse $(a_1x + b_1y + c_1)^2 + (a_2x + b_2y + c_2)^2 = 1$.

Sec. 9-6

1. The system of equations specifying projective transformation is uniquely resolvable to within a common factor with respect to the coefficients a_{ij} if four points and their images are specified.

2. Using the projective transformation, carry the points A, B, C into the points -1, 0, 1 on the x-axis and express all anharmonic ratios in terms of the coordinate ξ of the fourth point (D).

Sec. 9-7

1. Solve the system with respect to x_1', x_2', x_3'

$$\left.\begin{aligned} \lambda_1 x_1 &= a_1 x_1' + b_1 x_2' + c_1 x_3' \\ \lambda_2 x_2 &= a_2 x_1' + b_2 x_2' + c_2 x_3' \\ \lambda_3 x_3 &= a_3 x_1' + b_3 x_2' + c_3 x_3'. \end{aligned}\right\}$$

2. The straight lines intersect at point $(k_1, k_2, k_3, 0)$.

Sec. 9-8

For the first curve

$$x_1' = a_1x_1 + a_2x_2 + a_3x_3, \quad x_2' = b_1x_1 + b_2x_2 + b_3x_3, \quad x_3' = x_3$$

For the second curve

$$\begin{aligned} 2x_1' &= a_1x_1 + a_2x_2 + a_3x_3 + (b_1x_1 + b_2x_2 + b_3x_3), \\ 2x_2' &= a_1x_1 + a_2x_2 + a_3x_3 - (b_1x_1 + b_2x_2 + b_3x_3), \end{aligned} \quad x_3' = x_3.$$

Sec. 9-9

1. Compute the anharmonic ratio by passing over to the homogeneous coordinates.
2. Using the projective transformation, carry the straight line BH into an infinitely distant line.
3. Make use of the properties of a complete quadrilateral. See Exercise 2.
4. Draw an arbitrary straight line through the point of tangency. The polar lines of the points of this straight line intersect on the required tangent line.
5. The polar lines of two points of the straight line intersect at the required pole.
6. Compare the equation of the polar of the curve given by the general equation with the equation of the polars of the vertices of a self-polar triangle.
7. See Exercise 1.
8. Take the equation of the conic section in the canonical form and form the equation of the polar line of the focus.

Sec. 9-10

1. The projective transformation

$$x' = \frac{x}{ax+by+c}, \qquad y' = \frac{y}{ax+by+c}$$

preserves the bundle with centre at the origin but transforms its secants.

2. Make use of the correlative transformation of the plane.

Printed in the Union of Soviet Socialist Republics